EXCEPTIONAL COLLECTIONS ON DOLGACHEV SURFACES ASSOCIATED WITH DEGENERATIONS

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Abstract. Dolgachev surfaces are simply connected minimal elliptic surfaces with $p_g = q = 0$ and of Kodaira dimension 1. These surfaces were constructed by logarithmic transformations of rational elliptic surfaces. In this paper, we explain the construction of Dolgachev surfaces via $\mathbb{Q}$-Gorenstein smoothing of singular rational surfaces with two cyclic quotient singularities. This construction is based on the paper [22]. Also, some exceptional bundles on Dolgachev surfaces associated with $\mathbb{Q}$-Gorenstein smoothing are constructed based on the idea of Hacking [11]. In the case if Dolgachev surfaces were of type $(2,3)$, we describe the Picard group and present an exceptional collection of maximal length. Finally, we prove that the presented exceptional collection is not full, hence there exist a nontrivial phantom category in $\mathrm{D^b}(X^{\mathrm{g}})$.

Contents

1. Introduction

In the last few decades, the derived category $\mathrm{D^b}(S)$ of a nonsingular projective variety S has been extensively studied by algebraic geometers. One of the possible attempts is to find an exceptional collection that is a sequence of objects (mostly line bundles) $E_1, \ldots, E_n$ such that

$$\mathrm{Ext}^k(E_i, E_j) = \begin{cases} 0 & \text{if } i > j \\ 0 & \text{if } i = j \text{ and } k \neq 0 \\ \mathbb{C} & \text{if } i = j \text{ and } k = 0. \end{cases}$$

There were many approaches to find an exceptional collection of maximal length if S is a nonsingular projective surface with $p_g = q = 0$. Gorodentsev and Rudakov [10] classified all possible exceptional collection in the case $S = \mathbb{P}^2$, and exceptional collections on del Pezzo surfaces were studied by Kuleshov and Orlov [18]. For Enriques surfaces, Zube [33] found an exceptional collection of length 10, and the orthogonal part was studied by Ingalls and Kuznetsov [14] for nodal Enriques surfaces. After initiated by the work of Böhning, Graf von Bothmer, and Sosna [4], there also comes numerous results on the surfaces of general type (e.g. [1, 3, 6, 8, 9, 15, 20]). For the surfaces with Kodaira dimenion is one, such

exceptional collections are not known, thus it is a natural attempt to find an exceptional collection in $\mathrm{D^b}(S)$. In this paper, we use the technique of $\mathbb{Q}$-Gorenstein smoothing to study the case $\kappa(S) = 1$. As far as the authors know, this is the first time to establish an exceptional collection of maximal length on a surface with Kodaira dimension one.

The key ingredient is the method of Hacking [11], which associates a T_1-singularity $(P \in X)$ with an exceptional vector bundle on the general fiber of a $\mathbb{Q}$-Gorenstein smoothing of X. A T_1-singularity is a cyclic quotient singularity

$$(0 \in \mathbb{A}^2/\langle\xi\rangle), \quad \xi \cdot (x, y) = (\xi x, \xi^{na-1} y),$$

where $n > a > 0$ are coprime integers and ξ is the primitive n^2-th root of unity (see the works of Kollár and Shepherd-Barron [17], Manetti [23], and Wahl [31, 32] for the classification of T_1-singularities and their smoothings). In the paper [22], Lee and Park constructed new surfaces of general type via $\mathbb{Q}$-Gorenstein smoothings of projective normal surfaces with T_1-singularities. Motivated from [22], substantial amount of works were carried out, especially on (1) construction of new surfaces of general type (e.g. [16, 21, 26, 27]); (2) investigation of KSBA boundary of moduli of surfaces of general type (e.g. [12, 28]). Our approach is based on rather different persepctive:

Construct a smoothing $X \rightsquigarrow S$ using [22], and apply [11] to investigate $\operatorname{Pic} S$.

We study the case $S =$ a Dolgachev surface with two multiple fibers of multiplicities 2 and 3, and give an explicit $\mathbb{Z}$-basis for the Néron-Severi lattice of S (Theorem 1.2). Afterwards, we find an exceptional collection of line bundles of maximal length in $\mathrm{D^b}(S)$ (Theorem 1.4).

Notations and Conventions. Throughout this paper, everything will be defined over the field of complex numbers. A surface is an irreducible projective variety of dimension two. If T is a scheme of finite type over $\mathbb{C}$ and $t \in T$ a closed point, then we use $(t \in T)$ to indicate the analytic germ. This means that T is a small analytic neighborhood of t which can be shrunk if necessary.

Let $n > a > 0$ be coprime integers, and let ξ be the n^2-th root of unity. The T_1-singularity

$$(0 \in \mathbb{A}^2/\langle\xi\rangle), \quad \xi \cdot (x, y) = (\xi x, \xi^{na-1} y)$$

will be denoted by $\big(0 \in \mathbb{A}^2/\frac{1}{n^2}(1, na-1)\big)$.

By a divisor, we shall always mean a Cartier divisor unless stated otherwise. A divisor D is effective if $H^0(D) \neq 0$, namely, D is linearly equivalent to a nonnegative sum of integral divisors. If two divisors D_1 and D_2 are linearly equivalent, we write $D_1 = D_2$ if there is no ambiguity. Two $\mathbb{Q}$-Cartier Weil divisors D_1, D_2 are $\mathbb{Q}$-linearly equivalent, denoted by $D_1 \equiv D_2$, if there exists $n \in \mathbb{Z}_{>0}$ such that $nD_1 = nD_2$. We do not need an extra notion of numerical equivalence in this paper.

Let S be a nonsingular projective variety. The following invariants are associated with S.

- The geometric genus $p_g(S) = h^2(\mathcal{O}_S)$.
- The irregularity $q(S) = h^1(\mathcal{O}_S)$.
- The holomorphic Euler characteristic $\chi(S)$.
- The Néron-Severi group $\mathrm{NS}(S) = \operatorname{Pic} S/\operatorname{Pic}^0 S$, where $\operatorname{Pic}^0 S$ is the group of divisors algebraically equivalent to zero.

Since the definition of Dolgachev surfaces varies in literature, we fix our definition.

Definition 1.1. Let $q > p > 0$ be coprime integers. A *Dolgachev surface S of type (p,q)* is a minimal, simply connected, nonsingular, projective surface with $p_g(S) = q(S) = 0$ and of Kodaira dimension one such that there are exactly two multiple fibers of multiplicities p and q.

In the sequel, we will be given a degeneration $S \rightsquigarrow X$ from a nonsingular projective surface S to a projective normal surface X, and compare information between them. We use the superscript "g" to emphasize this correlation. For example, we use X^{g} instead of S. If $D \in \operatorname{Pic} X$ is a divisor that "deforms" to X^{g}, then the resulting divisor is denoted by D^{g}. However, usage of this convention will always be explcit; we explain the definition in each circumstance.

Synopsis of the paper. In Section 2, we construct a Dolgachev surface X^{g} of type $(2,n)$ following the technique of Lee and Park [22]. We begin with a pencil of plane cubics generated by two general nodal cubics, which meet nine different points. The pencil defines a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, undefined at the nine points of intersection. Blowing up the nine intersection points resolves the indeterminacy of $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, hence yields a rational elliptic surface. After additional blow ups, we get two special fibers

$$F_1 := C_1 \cup E_1, \quad \text{and} \quad F_2 := C_2 \cup E_2 \cup \ldots \cup E_{r+1}.$$

Let Y denote the resulting rational elliptic surface with the general fiber C_0, and let $p\colon Y \to \mathbb{P}^2$ denote the blow down morphism. Contracting the curves in the F_1 fiber (resp. F_2 fiber) except E_1 (resp. E_{r+1}), we get a morphism $\pi\colon Y \to X$ to a projective normal surface X with two T_1-singularities of types

$$(P_1 \in X) \simeq \Big(0 \in \mathbb{A}^2\Big/\frac{1}{4}(1,1)\Big) \quad \text{and} \quad (P_2 \in X) \simeq \Big(0 \in \mathbb{A}^2\Big/\frac{1}{n^2}(1,na-1)\Big)$$

for coprime integers $n > a > 0$. Note that the numbers n, a are determined by the formula

$$\frac{n^2}{na-1} = (-b_1) - \cfrac{1}{(-b_2) - \cfrac{1}{\cdots - \cfrac{1}{-b_r}}},$$

where $b_1, \ldots, b_r$ are the self-intersection numbers of the curves in the chain $\{C_2, \ldots, E_r\}$ (with the suitable order). We prove the formula (Proposition 2.2)

$$\pi^* K_X \equiv -C_0 + \frac{1}{2}C_0 + \frac{n-1}{n}C_0, \tag{1.1}$$

which resembles the canonical bundle formula for minimal elliptic surfaces [2, p. 213]. We then obtain X^{g} by taking a general fiber of a $\mathbb{Q}$-Gorenstein smoothing of X. Then, the divisor $\pi_* C_0$ is away from singularities of X, it moves to a nonsingular elliptic curve C_0^{g} along the deformation $X \rightsquigarrow X^{\mathrm{g}}$. We prove that the linear system $|C_0^{\mathrm{g}}|$ defines an elliptic fibration $f^{\mathrm{g}}\colon X^{\mathrm{g}} \to \mathbb{P}^1$. Comparing (1.1) with the canonical bundle formula on X^{g}, we achieve the following theorem.

Theorem 1.2 (see Theorem 2.8 for details)**.** *Let $\varphi\colon \mathcal{X} \to (0 \in T)$ be a one parameter $\mathbb{Q}$-Gorenstein smoothing of X over a smooth curve germ. Then for general $0 \neq t_0 \in T$, the fiber $X^{\mathrm{g}} := \mathcal{X}_{t_0}$ is a Dolgachev surface of type $(2,n)$.*

We jump into the case $a = 1$ in Section 3, and explain the constructions of exceptional bundles on X^{g} associated with the degeneration $X^{\mathrm{g}} \rightsquigarrow X$. For the construction of line bundles, we consider the short exact sequence (Proposition 3.2)

$$0 \to H_2(X^{\mathrm{g}}, \mathbb{Z}) \to H_2(X, \mathbb{Z}) \to H_1(M_1, \mathbb{Z}) \oplus H_1(M_2, \mathbb{Z}) \to 0$$

where M_i is the Milnor fiber of the smoothing of $(P_i \in X)$. Since $H_1(M_1,\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H_2(M_2,\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$, if $D \in \operatorname{Pic} Y$ is a divisor such that

$$(D.C_1) = 2d_1 \in 2\mathbb{Z},\ (D.C_2) = nd_2 \in n\mathbb{Z},\ \text{and } (D.E_2) = \ldots = (D.E_r) = 0, \tag{1.2}$$

then $[\pi_* D] \in H_2(X,\mathbb{Z})$ maps to the zero element in $H_1(M_1,\mathbb{Z}) \oplus H_1(M_2,\mathbb{Z})$. Thus, there exists a preimage $D^{\mathrm{g}} \in \operatorname{Pic} X^{\mathrm{g}}$ of $[\pi_* D] \in H_2(X,\mathbb{Z})$ along $\operatorname{Pic} X^{\mathrm{g}} \simeq H_2(X^{\mathrm{g}},\mathbb{Z}) \to H_2(X,\mathbb{Z})$. The next step is to investigate the relation between D and D^{g}. Let $\iota\colon Y \to \tilde{X}_0$ be the contraction of $E_2,\ldots,E_r$. Then, $Z_1 := \iota(C_1)$ and $Z_2 := \iota(C_2)$ are smooth rational curves. There exists a proper birational morphism $\Phi\colon \tilde{\mathcal{X}} \to \mathcal{X}$ (a weighted blow up at the singularities of $X = \mathcal{X}_0$) such that the central fiber $\tilde{\mathcal{X}}_0 := \Phi^{-1}(\varphi^{-1}(0))$ is described as follows: it is the union of $\tilde{X}_0$, the projective plane $W_1 = \mathbb{P}^2_{x_1,y_1,z_1}$, and the weighted projective plane $W_2 = \mathbb{P}_{x_2,y_2,z_2}(1,n-1,1)$ attached along

$$Z_1 \simeq (x_1y_1 = z_1^2) \subset W_1, \quad \text{and} \quad Z_2 \simeq (x_2y_2 = z_2^n) \subset W_2.$$

Intersection theory on W_1 and W_2 tells $\mathcal{O}_{W_1}(1)\big|_{Z_1} = \mathcal{O}_{Z_1}(2)$ and $\mathcal{O}_{W_2}(n-1)\big|_{Z_2} = \mathcal{O}_{Z_2}(n)$. The central fiber $\tilde{\mathcal{X}}_0$ has three irreducible components (disadvantage), but each component is more manageable than X (advantage). We work with the smoothing $\tilde{\mathcal{X}}/(0 \in T)$ instead of $\mathcal{X}/(0 \in T)$. The general fiber of $\tilde{\mathcal{X}}/(0 \in T)$ does not differ from $\mathcal{X}/(0 \in T)$, hence it is the Dolgachev surface X^{g}. If D is a divisor on Y satisfying (1.2), then there exists a line bundle $\tilde{\mathcal{D}}$ on $\tilde{\mathcal{X}}_0$ such that

$$\tilde{\mathcal{D}}\big|_{\tilde{X}_0} \simeq \mathcal{O}_{\tilde{X}_0}(\iota_* D), \quad \tilde{\mathcal{D}}\big|_{W_1} \simeq \mathcal{O}_{W_1}(d_1), \quad \text{and} \quad \tilde{\mathcal{D}}\big|_{W_2} \simeq \mathcal{O}_{W_2}((n-1)d_2).$$

Since the line bundle $\tilde{\mathcal{D}}$ is exceptional, it deforms uniquely to give a bundle $\mathscr{D}$ on the family $\tilde{\mathcal{X}}$. We define $D^{\mathrm{g}} \in \operatorname{Pic} X^{\mathrm{g}}$ to be the divisor associated with the line bundle $\mathscr{D}\big|_{X^{\mathrm{g}}}$.

Section 4 concerns the case $n = 3$ and $a = 1$. Let D, $\tilde{\mathcal{D}}$ and D^{g} be chosen as above. There exists a short exact sequence

$$0 \to \tilde{\mathcal{D}} \to \mathcal{O}_{\tilde{X}_0}(\iota_* D) \oplus \mathcal{O}_{W_1}(d_1) \oplus \mathcal{O}_{W_2}(2d_2) \to \mathcal{O}_{Z_1}(2d_1) \oplus \mathcal{O}_{Z_2}(3d_2) \to 0. \tag{1.3}$$

This expresses $\chi(\tilde{\mathcal{D}})$ in terms of $\chi(\iota_* D)$. Since Euler characteristic is a deformation invariant, we get $\chi(D^{\mathrm{g}}) = \chi(\tilde{\mathcal{D}})$. Furthermore, it can be proven that $(C_0.D) = (C_0^{\mathrm{g}}.D^{\mathrm{g}})$. This implies that $(C_0.D) = (6K_{X^{\mathrm{g}}}.D^{\mathrm{g}})$. The Riemann-Roch formula reads

$$(D^{\mathrm{g}})^2 = \frac{1}{6}(C_0.D) + 2\chi(\tilde{\mathcal{D}}) - 2,$$

which is a clue for discovering the Néron-Severi lattice $\operatorname{NS}(X^{\mathrm{g}})$. This leads to the first main theorem of this paper:

Theorem 1.3 (= Theorem 4.8)**.** *Let $H \in \operatorname{Pic}\mathbb{P}^2$ be the hyperplane divisor, and let $L_0 = p^*(2H)$. Consider the following correspondences of divisors (see Figure 2.1).*

$\operatorname{Pic} Y$	$F_i - F_j$	$p^*H - 3F_9$	L_0
$\operatorname{Pic} X^{\mathrm{g}}$	F_{ij}^{g}	$(p^*H - 3F_9)^{\mathrm{g}}$	L_0^{g}

Define the divisors $\{G_i^{\mathrm{g}}\}_{i=1}^{10} \subset \operatorname{Pic} X^{\mathrm{g}}$ as follows:

$$\begin{aligned} G_i^{\mathrm{g}} &= -L_0^{\mathrm{g}} + 10K_{X^{\mathrm{g}}} + F_{i9}^{\mathrm{g}}, \quad i = 1,\ldots,8; \\ G_9^{\mathrm{g}} &= -L_0^{\mathrm{g}} + 11K_{X^{\mathrm{g}}}; \\ G_{10}^{\mathrm{g}} &= -3L_0^{\mathrm{g}} + (p^*H - 3F_9)^{\mathrm{g}} + 28K_{X^{\mathrm{g}}}. \end{aligned}$$

Then the intersection matrix $\big((G_i^{\mathrm{g}}.G_j^{\mathrm{g}})\big)$ *is*

$$\begin{bmatrix} -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

In particular, $\{G_i^{\mathrm{g}}\}_{i=1}^{10}$ *is a* $\mathbb{Z}$*-basis for the Néron-Severi lattice* $\mathrm{NS}(X^{\mathrm{g}})$*.*

We point out that the assumption $n = 3$ is crucial for the definition of G_{10}^{g}. Indeed, its definition is motivated from the proof of [29, Theorem 3.1]. The divisor G_{10}^{g} was chosen to satisfy

$$K_{X^{\mathrm{g}}} = G_1^{\mathrm{g}} + \ldots + G_9^{\mathrm{g}} - 3G_{10}^{\mathrm{g}},$$

which does not valid for $n > 3$ as $K_{X^{\mathrm{g}}}$ is not primitive.

In Section 5 we continue to assume $n = 3$, $a = 1$. We give the proof of the second main theorem of the paper:

Theorem 1.4 (= Theorem 5.7 and Corollary 5.11)**.** *Assume that* X^{g} *is originated from a cubic pencil* $|\lambda p_* C_1 + \mu p_* C_2|$ *generated by two general nodal cubics. Then, there exists a semiorthogonal decomposition*

$$\big\langle \mathcal{A},\ \mathcal{O}_{X^{\mathrm{g}}},\ \mathcal{O}_{X^{\mathrm{g}}}(G_1^{\mathrm{g}}),\ \ldots,\ \mathcal{O}_{X^{\mathrm{g}}}(G_{10}^{\mathrm{g}}),\ \mathcal{O}_{X^{\mathrm{g}}}(2G_{10}^{\mathrm{g}})\big\rangle$$

of $\mathrm{D^b}(X^{\mathrm{g}})$*, where* $\mathcal{A}$ *is nontrivial phantom category (i.e.* $K_0(\mathcal{A}) = 0$*,* $\mathrm{HH}_\bullet(\mathcal{A}) = 0$*, and* $\mathcal{A} \not\simeq 0$*).*

The proof contains numerous cohomology computations. As usual, the main idea which relates the cohomologies between X and X^{g} is the upper-semicontinuity and the invariance of Euler characteristics. The cohomology long exact sequence of (1.3) begins with

$$0 \to H^0(\tilde{\mathcal{D}}) \to H^0(\iota_* D) \oplus H^0(\mathcal{O}_{W_1}(d_1)) \oplus H^0(\mathcal{O}_{W_2}(2d_2)) \to H^0(\mathcal{O}_{Z_1}(2d_1)) \oplus H^0(\mathcal{O}_{Z_2}(3d_2)).$$

We prove that if $(D.C_1) = 2d_1 \le 2$, $(D.C_2) = 3d_2 \le 3$, and $(D.E_2) = 0$, then $h^0(\tilde{\mathcal{D}}) \le h^0(D)$. This gives an upper bound of $h^0(D^{\mathrm{g}})$. By Serre duality, the upper bound of $h^2(D^{\mathrm{g}})$ can be carried out by observing $h^0(K_{X^{\mathrm{g}}} - D^{\mathrm{g}}) = 0$. After having upper bounds of $h^0(D^{\mathrm{g}})$ and $h^2(D^{\mathrm{g}}) = 0$, the upper bound of $h^1(D^{\mathrm{g}})$ can be examined by looking at $\chi(D^{\mathrm{g}})$. For any divisor D^{g} which appears in the proof of Theorem 1.4, at least one of $\{h^0(D^{\mathrm{g}}), h^2(D^{\mathrm{g}})\}$ is zero, and the other one is bounded by $\chi(D^{\mathrm{g}})$. Then, $h^1(D^{\mathrm{g}}) = 0$ and all the three numbers $(h^p(D^{\mathrm{g}}) : p = 0, 1, 2)$ are exactly evaluated. One obstruction to this argument is the condition $d_1, d_2 \le 1$, but it can be dealt with the following observation:

if a line bundle on X^{g} is obtained from C_1 or $2C_2 + E_2$, then it is trivial.

Perturbing D by C_1 and $2C_2 + E_2$, we can adjust the numbers d_1, d_2.

The proof reduced to find a suitable upper bound of $h^0(D)$. One of the very first trial is to find a smooth rational curve $C \subset Y$ such that $(D.C)$ is small. Then, by short exact sequence $0 \to \mathcal{O}_Y(D-C) \to \mathcal{O}_Y(D) \to \mathcal{O}_C(D) \to 0$, we get $h^0(D) \le h^0(D-C) + (C.D) + 1$. Replace D by $D - C$ and find another integral curve with small intersection. We repeat this procedure and stop when the value of $h^0(D-C)$ is understood immediately (e.g. when $D - C$ is linearly equivalent to a negative sum of effective curves). This will give an upper bound of the original D. This method sometimes gives a "sharp" upper bound of $h^0(D)$, but sometimes not. Indeed, some cohomologies depend on the configuration of generating cubics p_*C_1, p_*C_2 of the cubic pencil, while the previous numerical argument cannot capture the configuration

of p_*C_1 and p_*C_2. For those cases, we find an upper bound of $h^0(D)$ as follows. Assume that D is an effective divisor. Then, $p_*D \subset \mathbb{P}^2$ is a plane curve. The divisor form of D determines the degree of p_*D. Also, from the divisor form of p_*D, one can read the conditions that p_*D must admit. For example, consider $D = p^*H - E_1$. The exceptional curve E_1 is obtained by blowing up the node of p_*C_1. Hence, p_*D must be a line pass through the node of p_*C_1. In these ways, the imposed conditions can be represented by an ideal $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^2}$. Hence, proving $h^0(D) \leq r$ reduces to proving $h^0(\mathcal{O}_{\mathbb{P}^2}(\deg p_*D) \otimes \mathcal{I}) \leq r$. The latter one can be proved via a computer-based approach (Macaulay 2). Finally, $\mathcal{A} \not\simeq 0$ is guaranteed by the argument involving anticanonical pseudoheight due to Kuznetsov [19].

We remark that a (simply connected) Dolgachev surface of type $(2, n)$ cannot have an exceptional collection of maximal length for any $n > 3$ as explained in [29, Theorem 3.13]. Also, Theorem 1.4 give an answer to the question posed in [29, Remark 3.15].

Acknowledgements. The first author thanks to Kyoung-Seog Lee for helpful comments on derived categories. He also thanks to Alexander Kuznetsov for introducing the technique of height used in Section 5.2. The second author thanks to Fabrizio Catanese and Ilya Karzhemanov for useful remarks. This work is supported by Global Ph.D Fellowship Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(No.2013H1A2A1033339) (to Y.C.), and is partially supported by the NRF of Korea funded by the Korean government(MSIP)(No.2013006431) (to Y.L.).

2. Construction of Dolgachev Surfaces

Let n be an odd integer. This section presents a construction of Dolgachev surfaces of type $(2, n)$. The construction follows the technique introduced in [22]. Let $C_1, C_2 \subseteq \mathbb{P}^2$ be general nodal cubic curves meeting at 9 different points, and let $Y' = \operatorname{Bl}_9 \mathbb{P}^2 \to \mathbb{P}^2$ be the blow up at the intersection points. Then the cubic pencil $|\lambda C_1 + \mu C_2|$ defines an elliptic fibration $Y' \to \mathbb{P}^1$, with two special fibers C_1' and C_2' (which correspond to the proper transforms of C_1 and C_2, respectively). Blowing up the nodes of C_1' and C_2', we obtain (-1)-curves E_1, E_2. Also, blowing up one of the intersection points of C_2'' (the proper transform of C_2') and E_2, we obtain the configuration described in Figure 2.1. The divisors $F_1, \ldots, F_9$

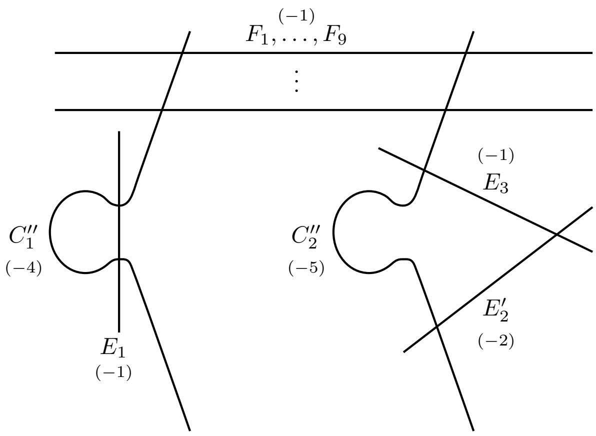
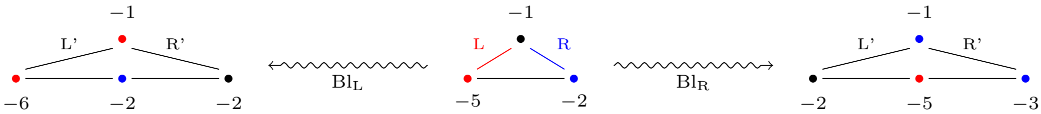


Figure 2.1. Configuration of the divisors in the surface obtained by blowing up two points of Y'.

are proper transforms of the exceptional fibers of the blow up $Y' = \operatorname{Bl}_9 \mathbb{P}^2 \to \mathbb{P}^2$. The numbers in the parentheses are self-intersection numbers of the corresponding divisors. On the fiber $C_2'' \cup E_2' \cup E_3$, we

can think of two different blow ups as the following dual intersection graphs illustrate.



In general, if one has a fiber with configuration $\underset{-k_1}{\bullet}$ — $\underset{-k_2}{\bullet}$ — $\cdots$ — $\underset{-k_r}{\bullet}$, then after the blowing up at L the graph becomes $\underset{-(k_1+1)}{\bullet}$ — $\underset{-k_2}{\bullet}$ — $\cdots$ — $\underset{-k_r}{\bullet}$ — $\underset{-2}{\bullet}$. Similarly, the blowing up at R yields the configuration $\underset{-2}{\bullet}$ — $\underset{-k_1}{\bullet}$ — $\cdots$ — $\underset{-k_{r-1}}{\bullet}$ — $\underset{-(k_r+1)}{\bullet}$. This presents all possible resolution graphs of T_1-singularities [23, Thm. 17]. Let Y be the surface after successive blow ups on the second special fiber $C_2'' \cup E_2' \cup E_3$, so that the resulting fiber contains the resolution graph of a T_1-singularity of type $\big(0 \in \mathbb{A}^2/\frac{1}{n^2}(1, na-1)\big)$ for some odd integer n and an integer a with $\gcd(n,a)=1$.

To simplify notations, we would not distinguish the divisors and their proper transforms unless they arise ambiguities. For instance, the proper transform of $C_1 \in \operatorname{Pic} \mathbb{P}^2$ in Y will be denoted by C_1, and so on. We fix this configuration of Y throughout this paper, so it is appropriate to give a summary here:

(1) the (-1)-curves $F_1, \ldots, F_9$ that are proper transforms of the exceptional fibers of $\mathrm{Bl}_9 \mathbb{P}^2 \to \mathbb{P}^2$;
(2) the (-4)-curve C_1 and the (-1)-curve E_1 arising from the blowing up of the first nodal curve;
(3) the negative curves C_2, E_2, …, E_r, E_{r+1}, where $E_{r+1}^2 = -1$ and C_2, E_2, …, E_r form a resolution graph of a T_1-singularity of type $\big(0 \in \mathbb{A}^2/\frac{1}{n^2}(1, na-1)\big)$.

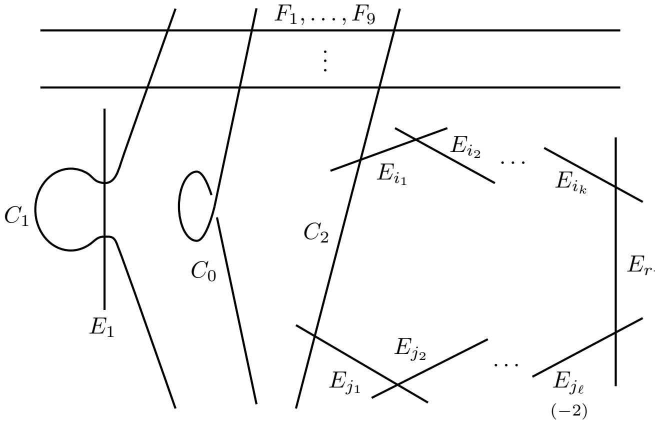


Figure 2.2. Configuration of the surface Y. The sequence $E_{i_k}, \ldots, E_{i_1}, C_2, E_{j_1}, \ldots, E_{j_\ell}$ forms a chain of the resolution graph of T_1-singularity of type $\big(0 \in \mathbb{A}^2/\frac{1}{n^2}(1, na-1)\big)$. Without loss of generality, we may assume $j_\ell = r$. Note that $E_r^2 = -2$ by construction.

Let C_0 be a general fiber of the elliptic fibration $Y \to \mathbb{P}^1$. The fibers are linearly equivalent, thus

$$\begin{aligned} C_0 &= C_1 + 2E_1 \\ &= C_2 + a_2E_2 + a_3E_3 + \ldots + a_{r+1}E_{r+1}, \end{aligned} \tag{2.4}$$

where $a_2, \ldots, a_{r+1}$ are the integers determined by the system of linear equations

$$(C_2.E_i) + \sum_{j=2}^{r+1} a_j (E_j.E_i) = 0, \quad i = 2, \ldots, r+1. \tag{2.5}$$

Note that the values $(C_2.E_i)$, $(E_j.E_i)$ are explicitly determined by the configuration (Figure 2.2). The matrix $\big((E_j.E_i)\big)_{2\le i,j\le r}$ is negative definite [24], and the number a_{r+1} is determined by Proposition 2.3, hence the system (2.5) has a unique solution.

Lemma 2.1. *In the above situation, the following formula holds:*

$$K_Y = E_1 - C_2 - E_2 - \ldots - E_{r+1}.$$

Proof. The proof proceeds by an induction on r. The minimum value of r is two, the case in which $C_2 \cup E_2$ from the chain $\overset{-5}{\bullet}\!\!-\!\!\!-\!\!\overset{-2}{\bullet}$. Let $H \in \operatorname{Pic}\mathbb{P}^2$ be a hyperplane divisor, and let $p\colon Y \to \mathbb{P}^2$ be the blowing down morphism. Then

$$K_Y = p^*K_{\mathbb{P}^2} + F_1 + \ldots + F_9 + E_1 + d_2E_2 + d_3E_3$$

for some $d_2, d_3 \in \mathbb{Z}$. Since any cubic curve in $\mathbb{P}^2$ is linearly equivalent to $3H$,

$$\begin{aligned} p^*(3H) &= C_0 + F_1 + \ldots + F_9 \\ &= (C_2 + a_2E_2 + a_3E_3) + F_1 + \ldots + F_9 \end{aligned}$$

where a_2, a_3 are integers introduced in (2.4). Hence,

$$\begin{aligned} K_Y &= p^*(-3H) + F_1 + \ldots + F_9 + E_1 + d_2E_2 + d_3E_3 \\ &= E_1 - C_2 + (d_2 - a_2)E_2 + (d_3 - a_3)E_3. \end{aligned}$$

Here, the genus formula shows that $K_Y = E_1 - C_2 - E_2 - E_3$. Assume the induction hypothesis that $K_Y = E_1 - C_2 - E_2 - \ldots - E_{r+1}$. Let $D \in \{C_2, E_2, \ldots, E_r\}$ be a divisor intersects E_{r+1}, and let $\varphi\colon \widetilde{Y} \to Y$ be the blowing up at the point $D \cap E_{r+1}$. Then,

$$K_{\widetilde{Y}} = \varphi^*K_Y + \widetilde{E}_{r+2},$$

where $\widetilde{E}_{r+2}$ is the exceptional divisor of the blowing up φ. Let $\widetilde{C}_2, \widetilde{E}_1, \ldots, \widetilde{E}_{r+1}$ denote the proper transforms of the corresponding divisors. Then, φ^* maps D to $(\widetilde{D} + \widetilde{E}_{r+2})$, maps E_{r+1} to $(\widetilde{E}_{r+1} + \widetilde{E}_{r+2})$, and maps the other divisors to their proper transforms. It follows that

$$\begin{aligned} \varphi^*K_Y &= \varphi^*(E_1 - C_2 - \ldots - E_{r+1}) \\ &= \widetilde{E}_1 - \widetilde{C}_2 - \ldots - \widetilde{E}_{r+1} - 2\widetilde{E}_{r+2}. \end{aligned}$$

Hence, $K_{\widetilde{Y}} = \varphi^*K_Y + \widetilde{E}_{r+2} = \widetilde{E}_1 - \widetilde{C}_2 - \widetilde{E}_2 - \ldots - \widetilde{E}_{r+2}$. □

Proposition 2.2. *Let $\pi\colon Y \to X$ be the contraction of the curves C_1, C_2, E_2, …, E_r. Let $P_1 = \pi(C_1)$ and $P_2 = \pi(C_2 \cup E_2 \cup \ldots \cup E_r)$ be the singularities of types $\big(0 \in \mathbb{A}^2/\frac{1}{4}(1,1)\big)$ and $\big(0 \in \mathbb{A}^2/\frac{1}{n^2}(1, na-1)\big)$, respectively. Then the following properties of X hold:*

(a) *X is a projective normal surface with $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$;*
(b) *$\pi^*K_X \equiv (\frac{1}{2} - \frac{1}{n})C_0 \equiv C_0 - \frac{1}{2}C_0 - \frac{1}{n}C_0$ as $\mathbb{Q}$-divisors.*

In particular, $K_X^2 = 0$, K_X is nef, but K_X is not numerically trivial.

Proof.

(a) Since the singularities of X are rational, $R^q\pi_*\mathcal{O}_Y = 0$ for $q > 0$. The Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{O}_Y) \Rightarrow H^{p+q}(Y, \mathcal{O}_Y)$$

says that $H^p(Y, \mathcal{O}_Y) \simeq H^p(X, \pi_*\mathcal{O}_Y) = H^p(X, \mathcal{O}_X)$ for $p > 0$. The surface Y is obtained from $\mathbb{P}^2$ by a finite sequence of blow ups, hence $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$.

(b) Since the morphism π contracts C_1, C_2, E_2, $\ldots$, E_r, we may write

$$\pi^*K_X \equiv K_Y + c_1C_1 + c_2C_2 + b_2E_2 + \ldots + b_rE_r,$$

for $c_1, c_2, b_2, \ldots, b_r \in \mathbb{Q}$(the coefficients may not integral since X is singular). It is easy to see that $c_1 = \frac{1}{2}$. By Lemma 2.1,

$$\pi^*K_X \equiv \frac{1}{2}C_0 + (c_2 - 1)C_2 + (b_2 - 1)E_2 + \ldots + (b_r - 1)E_r - E_{r+1}.$$

Both π^*K_X and C_0 do not intersect with $C_2, E_2, \ldots, E_r$. Thus, we get

$$\begin{cases} 0 = (1 - c_2)(C_2^2) + \sum_{j=2}^r (1 - b_j)(E_j.C_2) + (E_{r+1}.C_2) \\ 0 = (1 - c_2)(C_2.E_i) + \sum_{j=2}^r (1 - b_j)(E_j.E_i) + (E_{r+1}.E_i), \quad \text{for } i = 2, \ldots, r. \end{cases} \tag{2.6}$$

After divided by a_{r+1}, (2.5) becomes

$$0 = \frac{1}{a_{r+1}}(C_2.E_i) + \sum_{j=2}^r \frac{a_j}{a_{r+1}}(E_j.E_i) + (E_{r+1}.E_i), \quad \text{for } i = 2, \ldots, r.$$

In addition, the equation $(C_2 + a_2E_2 + \ldots + a_{r+1}E_r \,.\, C_2) = (C_0.C_2) = 0$ gives rise to

$$0 = \frac{1}{a_{r+1}}(C_2^2) + \sum_{j=2}^r \frac{a_j}{a_{r+1}}(E_j.C_2) + (E_{r+1}.C_2).$$

Comparing these equations with (2.6), it is easy to see that the ordered tuples

$$(1 - c_2,\ 1 - b_2,\ \ldots,\ 1 - b_r) \quad \text{and} \quad (1/a_{r+1},\ a_2/a_{r+1},\ \ldots,\ a_r/a_{r+1})$$

fit into the same system of linear equations. Since the intersection matrix of the divisors $(C_2, E_2, \ldots, E_r)$ is negative definite,

$$(1 - c_2, 1 - b_2, \ldots, 1 - b_r) = (1/a_{r+1},\ a_2/a_{r+1},\ \ldots,\ a_r/a_{r+1}).$$

It follows that

$$\begin{aligned} \pi^*K_X &\equiv \frac{1}{2}C_0 + (c_2 - 1)C_2 + (b_2 - 1)E_2 + \ldots + (b_r - 1)E_r - E_{r+1} \\ &\equiv \frac{1}{2}C_0 - \frac{1}{a_{r+1}}\big(C_2 + a_2E_2 + \ldots + a_{r+1}E_{r+1}\big) \\ &\equiv \Big(\frac{1}{2} - \frac{1}{a_{r+1}}\Big)C_0. \end{aligned}$$

It remains to prove $a_{n+1} = n$. This directly follows from Proposition 2.3. It is immediate to see that $C_0^2 = 0$, C_0 is nef, and C_0 is not numerically trivial. The same properties are true for π^*K_X. □

Proposition 2.3. *Suppose that $C_2 \cup E_2 \cup \ldots \cup E_r$ has the configuration* $\underset{-k_1}{\bullet}$ —— $\underset{-k_2}{\bullet}$ —— $\cdots$ —— $\underset{-k_r}{\bullet}$ *, so that it contracts to give a T_1-singularity of type $\big(0 \in \mathbb{A}^2 / \frac{1}{n^2}(1, na - 1)\big)$. Then, in the expression*

$$C_2 + a_2E_2 + \ldots + a_{r+1}E_{r+1}$$

of the fiber (2.4), the coefficient of the $(-k_1)$-curve is a, and the coefficient of the $(-k_r)$-curve is $(n-a)$. Furthermore, a_{r+1} equals to the sum of these two coefficients, hence $a_{r+1} = n$.

Proof. The proof proceeds by an induction on r. The case $r = 2$ is trivial. Indeed, a simple computations shows that $n = 3$, $a = 1$, and $a_2 = 2$, $a_3 = 3$. To make notations simpler, we reindex $\{C_2, E_2, \ldots, E_{r+1}\}$ as follows:

$$(G_1, G_2, \ldots, G_{r+1}) = (E_{i_k}, E_{i_{k-1}}, \ldots, E_{i_1}, C_2, E_{j_1}, \ldots, E_{j_\ell}, E_{r+1}). \qquad \text{(Figure 2.2)}$$

By the induction hypothesis, we may assume

$$C_2 + a_2 E_2 + \ldots + a_{r+1} E_{r+1} = aG_1 + \ldots + (n-a)G_r + nG_{r+1}.$$

Let $\varphi_1 \colon \widetilde{Y} \to Y$ be the blow up at the point $G_{r+1} \cap G_1$, let $\widetilde{G}_i$ $(i = 1, \ldots, r+1)$ be the proper transform of G_i, and let $\widetilde{G}_{r+2}$ be the exceptional divisor. The (-1)-curve $\widetilde{G}_{r+2}$ meets $\widetilde{G}_1$ and $\widetilde{G}_{r+1}$ transversally, so

$$\begin{aligned}\varphi^*(aG_1 + \ldots + nG_{r+1}) &= a(\widetilde{G}_1 + \widetilde{G}_{r+2}) + g_2\widetilde{G}_2 + \ldots + (n-a)\widetilde{G}_r + n(\widetilde{G}_{r+1} + \widetilde{G}_{r+2}) \\ &= a\widetilde{G}_1 + g_2\widetilde{G}_2 + \ldots + (n-a)\widetilde{G}_r + n\widetilde{G}_{r+1} + (n+a)\widetilde{G}_{r+2}.\end{aligned}$$

It is well-known that the contraction of $\widetilde{G}_1, \ldots, \widetilde{G}_{r+1} \subset \widetilde{Y}$ produces a cyclic quotient singularity of type

$$\left(0 \in \mathbb{A}^2 \Big/ \frac{1}{(n+a)^2}(1, n(n+a)-1)\right).$$

This proves the statement for the chain $\widetilde{G}_1 \cup \ldots \cup \widetilde{G}_{r+2}$, so we are done by induction. The same argument also works if one performs the blow up $\varphi_2 \colon \widetilde{Y}' \to Y$ at the point $G_{r+1} \cap G_r$. □

Now we want to dissolve the singularities of X by $\mathbb{Q}$-Gorenstein smoothings. It is well-known that T_1-singularities admit local $\mathbb{Q}$-Gorenstein smoothings, thus we have to verify:

(a) every formal deformation of X is algebraizable;

(b) every local deformation of X can be globalized.

The answer for (a) is an immediate consequence of Grothendieck's existence theorem [13, Example 21.2.5] since $H^2(\mathcal{O}_X) = 0$. The next lemma verifies (b).

Lemma 2.4. *Let Y be the nonsingular rational elliptic surface introduced above, and let $\mathcal{T}_Y$ be the tangent sheaf of Y. Then,*

$$H^2(Y, \mathcal{T}_Y(-C_1 - C_2 - E_2 - \ldots - E_r)) = 0.$$

In particular, $H^2(X, \mathcal{T}_X) = 0$ (see [22, Thm. 2]).

Proof. The proof is not very different from [22, §4, Example 2]. The main claim is

$$H^0(Y, \Omega^1_Y(K_Y + C_1 + C_2 + E_2 + \ldots + E_r)) = 0.$$

By Lemma 2.1 and equation (2.4),

$$K_Y + C_1 + C_2 + E_2 + \ldots + E_r = C_0 - E_1 - E_{r+1}.$$

Then, $h^0(Y, \Omega^1_Y(C_0 - E_1 - E_{r+1})) \leq h^0(Y, \Omega^1_Y(C_0)) = h^0(Y', \Omega^1_{Y'}(C_0'))$ where $Y' = \mathrm{Bl}_9\,\mathbb{P}^2$, and $h^0(Y', \Omega^1_{Y'}(C_0')) = 0$ by [22, §4, Lemma 2]. The result directly follows from the Serre duality. □

We showed that the surface X admits a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X} \to T$. The next aim is to show that the general fiber $X^{\mathrm{g}} := \mathcal{X}_t$ is a Dolgachev surface of type $(2, n)$.

Proposition 2.5. *Let X be a projective normal surface with only rational singularities, let $\pi\colon Y \to X$ be a resolution of singularities, and let $E_1, \dots, E_r$ be the exceptional divisors. If D is a divisor on Y such that $(D.E_i) = 0$ for all $i = 1, \dots, r$, then*

$$H^p(Y, D) \simeq H^p(X, \pi_* D)$$

for all $p \geq 0$.

Proof. Since the singularities of X are rational, each E_i is a smooth rational curve. The assumption on D in the statement implies that $\pi_* D$ is Cartier, and $\pi^* \mathcal{O}_X(\pi_* D) = \mathcal{O}_Y(D)$. By projection formula, $R^p\pi_*\mathcal{O}_Y(D) \simeq R^p\pi_*(\mathcal{O}_Y \otimes \pi^*\mathcal{O}_X(\pi_* D)) \simeq (R^p\pi_*\mathcal{O}_Y) \otimes \mathcal{O}_X(\pi_* D)$. Since X is normal and has only rational singularities,

$$R^p\pi_*\mathcal{O}_Y = \begin{cases} \mathcal{O}_X & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

Now, the claim is an immediate consequence of the Leray spectral sequence

$$E_2^{p,q} = H^p(X,\, R^q\pi_*\mathcal{O}_Y \otimes \mathcal{O}_X(\pi_* D)) \Rightarrow H^{p+q}(Y,\, \mathcal{O}_Y(D)).$$ □

Lemma 2.6. *Let $\pi\colon Y \to X$ be the contraction defined in Proposition 2.2. Then,*

$$h^0(X, \pi_* C_0) = 2, \quad h^1(X, \pi_* C_0) = 1, \ \textit{and} \quad h^2(X, \pi_* C_0) = 0.$$

Proof. It is easy to see that $(C_0.C_1) = (C_0.C_2) = (C_0.E_2) = \ldots = (C_0.E_r) = 0$. Hence by Proposition 2.5, it suffices to compute $h^p(Y, C_0)$. Since $C_0^2 = (K_Y.C_0) = 0$, Riemann-Roch formula shows $\chi(C_0) = 1$. By Serre duality, $h^2(C_0) = h^0(K_Y - C_0)$. In the short exact sequence

$$0 \to \mathcal{O}_Y(K_Y - C_0 - E_1) \to \mathcal{O}_Y(K_Y - C_0) \to \mathcal{O}_{E_1} \otimes \mathcal{O}_Y(K_Y - C_0) \to 0,$$

we find that $H^0(\mathcal{O}_{E_1} \otimes \mathcal{O}_Y(K_Y - C)) = 0$ since $(K_Y - C_0 \,.\, E_1) = -1$. It follows that

$$h^0(K_Y - C_0) = h^0(K_Y - C_0 - E_1),$$

but $K_Y - C_0 - E_1 = -2C_2 - (a_2+1)E_2 - \ldots - (a_{r+1}+1)E_{r+1}$ by Lemma 2.1. Hence $h^2(C_0) = 0$. Since the complete linear system $|C_0|$ defines the elliptic fibration $Y \to \mathbb{P}^1$, $h^0(C_0) = 2$. Furthermore, $h^1(C_0) = 1$ follows from $h^0(C_0) = 2$, $h^2(C_0) = 0$, and $\chi(C_0) = 1$. □

The following proposition, due to Manetti [23], is a key ingredient of the proof of Theorem 2.8

Proposition 2.7 ([23, Lemma 2])**.** *Let $\mathcal{X} \to (0 \in T)$ be a smoothing of a normal surface X with $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$. Then for every $t \in T$, the natural restriction map of second cohomology groups $H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}_t, \mathbb{Z})$ induces an injection $\operatorname{Pic}\mathcal{X} \to \operatorname{Pic}\mathcal{X}_t$. Furthermore, the restriction to the central fiber $\operatorname{Pic}\mathcal{X} \to \operatorname{Pic} X$ is an isomorphism.*

Theorem 2.8. *Let X be the projective normal surface defined in Proposition 2.2, and let $\varphi\colon \mathcal{X} \to (0 \in T)$ be a one parameter $\mathbb{Q}$-Gorenstein smoothing of X over a smooth curve germ $(0 \in T)$. For general $0 \neq t_0 \in T$, the fiber $X^{\mathrm{g}} := \mathcal{X}_{t_0}$ satisfies the following:*

(a) $p_g(X^{\mathrm{g}}) = q(X^{\mathrm{g}}) = 0$;
(b) *X^{g} is a simply connected, minimal, nonsingular surface with Kodaira dimension* 1;
(c) *there exists an elliptic fibration $f^{\mathrm{g}}\colon X^{\mathrm{g}} \to \mathbb{P}^1$ such that $K_{X^{\mathrm{g}}} \equiv C_0^{\mathrm{g}} - \frac{1}{2}C_0^{\mathrm{g}} - \frac{1}{n}C_0^{\mathrm{g}}$, where C_0^{g} is a general nonsingular elliptic fiber of f^{g};*
(d) *X^{g} is isomorphic to the Dolgachev surface of type* $(2, n)$.

Proof.

(a) This follows from Proposition 2.2(a) and the upper-semicontinuity of h^p.

(b) Shrinking $(0 \in T)$ if necessary, we may assume that X^{g} is simply connected [22, p. 499], and that $K_{X^{\mathrm{g}}}$ is nef [25, §5.d]. If $K_{X^{\mathrm{g}}}$ is numerically trivial, then X^{g} must be an Enriques surface by classification of surfaces. This violates the simple connectivity of X^{g}. It follows that $K_{X^{\mathrm{g}}}$ is not numerically trivial, and the Kodaira dimension of X^{g} is 1.

(c) Since the divisor $\pi_* C_0$ is not supported on the singular points of X, $\pi_* C_0 \in \operatorname{Pic} X$. By Proposition 2.7, $\operatorname{Pic} X \simeq \operatorname{Pic} \mathcal{X} \hookrightarrow \operatorname{Pic} X^{\mathrm{g}}$. Let $C_0^{\mathrm{g}} \in \operatorname{Pic} X^{\mathrm{g}}$ be the image of $\pi_* C_0$ under this correspondence. In Section 3.1, we will see that there exists divisors E_1^{g} (resp. E_{r+1}^{g}), which maps to $\pi_* E_1$ (resp. $\pi_* E_{r+1}$) along the specialization map $\operatorname{Pic} X^{\mathrm{g}} \hookrightarrow \operatorname{Cl} X$. Clearly, $2E_1^{\mathrm{g}}$ are nE_{r+1}^{g} are different as closed subschemes, however, both $2E_1^{\mathrm{g}}$ and nE_{r+1}^{g} are linearly equivalent to C_0^{g} since $\pi_*(2E_1) = \pi_* C_0 = \pi_*(nE_{r+1})$. It follows that $h^0(C_0^{\mathrm{g}}) \geq 2$. By upper-semicontinuity, Proposition 2.5, and Lemma 2.6, $h^0(C_0^{\mathrm{g}}) = 2$.

By the $\mathbb{Q}$-Gorenstein condition of $\mathcal{X}/(0 \in T)$, $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. Since $2nK_X = (n-2)\pi_* C_0$ is Cartier, the isomorphism $\operatorname{Pic} X \simeq \operatorname{Pic} \mathcal{X}$ maps $2nK_X$ to the Cartier divisor $2nK_{\mathcal{X}}$. This shows that the map $\operatorname{Pic} X \hookrightarrow \operatorname{Pic} X^{\mathrm{g}}$ sends $2nK_X$ to $2nK_{X^{\mathrm{g}}}$. Furthermore, $2nK_X - (n-2)\pi_* C_0 \in \operatorname{Pic} X$ is trivial and it maps to $2nK_{X^{\mathrm{g}}} - (n-2)C_0^{\mathrm{g}}$, hence

$$K_{X^{\mathrm{g}}} \equiv C_0^{\mathrm{g}} - \frac{1}{2} C_0^{\mathrm{g}} - \frac{1}{n} C_0^{\mathrm{g}}.$$

This shows that $(K_X^{\mathrm{g}} . C_0^{\mathrm{g}}) = \frac{2n}{n-2} K_{X^{\mathrm{g}}}^2 = \frac{2n}{n-2} K_X^2 = 0$. Furthermore, since $\chi(C_0^{\mathrm{g}}) = \chi(\pi_* C_0) = \chi(C_0) = 1$, we get $(C_0^{\mathrm{g}})^2 = 0$.

Now, we claim that the complete linear system $|C_0^{\mathrm{g}}|$ is base point free; indeed, if $p \in |C_0^{\mathrm{g}}|$ is a base point, then two different closed subschemes $2E_1^{\mathrm{g}}, nE_{r+1}^{\mathrm{g}} \in |C_0^{\mathrm{g}}|$ intersect at p, thus $(2E_1^{\mathrm{g}} \,.\, nE_{r+1}^{\mathrm{g}}) = (C_0^{\mathrm{g}})^2 > 0$, a contradiction. It follows that the linear system $|C_0^{\mathrm{g}}|$ defines an elliptic fibration $f^{\mathrm{g}} \colon X^{\mathrm{g}} \to \mathbb{P}^1$ with the general fiber C_0^{g}.

(d) By [7, Chapter 2], every minimal simply connected nonsingular surface with $p_g = q = 0$ and of Kodaira dimension 1 has exactly two multiple fibers with coprime multiplicities. Thus, there exist coprime integers $q > p > 0$ such that $X^{\mathrm{g}} \simeq X_{p,q}$ where $X_{p,q}$ is a Dolgachev surface of type (p, q). The canonical bundle formula says that $K_{X_{p,q}} \equiv C_0^{\mathrm{g}} - \frac{1}{p} C_0^{\mathrm{g}} - \frac{1}{q} C_0^{\mathrm{g}}$. Since $X^{\mathrm{g}} \simeq X_{p,q}$, this leads to the equality

$$\frac{1}{2} + \frac{1}{n} = \frac{1}{p} + \frac{1}{q}.$$

Assume $2 < p < q$. Then, $\frac{1}{2} < \frac{1}{2} + \frac{1}{n} = \frac{1}{p} + \frac{1}{q} \leq \frac{1}{3} + \frac{1}{q}$. Hence, $q < 6$. Only the possible candidates are $(p, q, n) = (3, 4, 12)$, $(3, 5, 30)$, but all of these cases violate $\gcd(2, n) = 1$. It follows that $p = 2$ and $q = n$. $\square$

Remark 2.9. Theorem 2.8 generalizes to constructions of Dolgachev surfaces of type (m, n) for any coprime integers $n > m > 0$. Indeed, as mentioned in the proof, we shall see that the Weil divisor $\pi_* E_{r+1}$ deforms to the multiple fiber of multiplicity n (see Example 5.3). If we perform more blow ups to the $C_1 \cup E_1$ fiber so that X has a T_1-singularity of type $\left(0 \in \mathbb{A}^2 \big/ \frac{1}{m^2}(1, mb-1)\right)$, then the surface X^{g} has two multiple fibers of multiplicites m and n. Thus, X^{g} is a Dolgachev surface of type (m, n).

3. Exceptional vector bundles on Dolgachev surfaces

In general, it is hard to understand how information of the central fiber is carried to the general fiber along a $\mathbb{Q}$-Gorenstein smoothing. Looking at the topology nearby the singularities of X, one can get a clue to relate information between X and X^{g}. This section essentially follows the idea of Hacking. Some ingredients of Hacking's method, which are necessary for our application, are included in the appendix(Section 6). Readers who want to look up details are recommended to consult Hacking's original paper [11].

3.1. **Topology of the singularities of X.** Let $L_i \subseteq X\,(i = 1, 2)$ be the link of the singularity P_i. Then, $H_1(L_1, \mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z}$ and $H_1(L_2, \mathbb{Z}) \simeq \mathbb{Z}/n^2\mathbb{Z}$ (cf. [23, Proposition 13]). Since $\gcd(2, n) = 1$, $H_1(L_1, \mathbb{Z}) \oplus H_1(L_2, \mathbb{Z}) \simeq \mathbb{Z}/4n^2\mathbb{Z}$ is a finite cyclic group. By [11, p. 1191], $H_2(X, \mathbb{Z}) \to H_1(L_i, \mathbb{Z})$ is surjective for each $i = 1, 2$, thus the natural map

$$H_2(X, \mathbb{Z}) \to H_1(L_1, \mathbb{Z}) \oplus H_1(L_2, \mathbb{Z}), \quad \alpha \mapsto (\alpha \cap L_1,\, \alpha \cap L_2)$$

is surjective. We have further information on groups $H_1(L_i, \mathbb{Z})$.

Theorem 3.1 ([24])**.** *Let X be a projective normal surface containing a T_1-singularity $P \in X$. Let $f\colon \widetilde{X} \to X$ be a good resolution (i.e. the exceptional divisor is simple normal crossing) of the singularity P, and let $E_1, \ldots, E_r$ be a chain of exceptional divisors such that $(E_i.E_{i+1}) = 1$ for each $i = 1, \ldots, r-1$. Let $\widetilde{L} \subseteq \widetilde{X}$ be the plumbing fixture (see Figure 3.3) around $\bigcup E_i$, and let $\alpha_i \subset \widetilde{L}$ be the loop around E_i oriented suitably. Then the following statements are true.*

(a) *The group $H_1(\widetilde{L}, \mathbb{Z})$ is generated by the loops α_i. The relations are*

$$\sum_j (E_i.E_j)\alpha_j = 0, \quad i = 1, \ldots, r.$$

(b) *Let $L \subset X$ be the link of the singularity $P \in X$. Then, $\widetilde{L}$ is homeomorphic to L.*

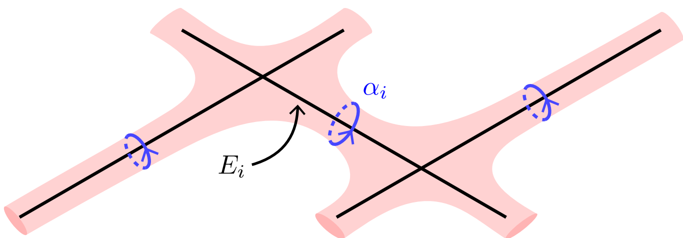


Figure 3.3. Plumbing fixture around $\bigcup E_i$.

Proposition 2.7 provides a way to associate a Cartier divisor on X with a Cartier divisor on X^{g}. This association can be extended as the following proposition illustrates.

Proposition 3.2 ([11, Lemma 5.5])**.** *Let X be a projective normal surface, and let $(P \in X)$ be a T_1-singularity of type $\left(0 \in \mathbb{A}^2/\frac{1}{n^2}(1, na-1)\right)$. Suppose X admits a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}/(0 \in T)$ over a smooth curve germ $(0 \in T)$ such that $\mathcal{X}/(0 \in T)$ is a smoothing of $(P \in X)$, and is locally trivial outside $(P \in X)$. Let X^{g} be a general fiber of $\mathcal{X} \to (0 \in T)$, and let $\mathcal{B} \subset \mathcal{X}$ be a sufficiently small open ball around $P \in \mathcal{X}$. Then the link L and the Milnor fiber M of $(P \in X)$ given as follows:*

$$L = \partial\mathcal{B} \cap X^{\mathrm{g}}, \qquad M = \mathcal{B} \cap X^{\mathrm{g}}.$$

In addition, let $B := \mathcal{B} \cap X$ be the contractible space [11, §7.1]. *Assume that X^{g} is simply connected, $H^2(\mathcal{O}_{X^{\mathrm{g}}}) = 0$, and the natural map $H_2(X,\mathbb{Z}) \to H_1(L,\mathbb{Z})$ (the connecting homomorphism of the Mayer-Vietoris sequence associated to the decomposition $X = (X \setminus B) \cup B$) is surjective. Then, there exists a short exact sequence*

$$0 \to H_2(X^{\mathrm{g}},\mathbb{Z}) \to H_2(X,\mathbb{Z}) \to H_1(M,\mathbb{Z}) \to 0.$$

Here, the specialization map $H_2(X^{\mathrm{g}},\mathbb{Z}) \to H_2(X,\mathbb{Z})$ is defined by the composition

$$H_2(X^{\mathrm{g}}) \simeq H^2(X^{\mathrm{g}}) \to H^2(X^{\mathrm{g}} \setminus M) \simeq H^2(X \setminus B) \simeq H_2(X \setminus B, L) \simeq H_2(X,B) \simeq H_2(X),$$

and $H_2(X,\mathbb{Z}) \to H_1(M,\mathbb{Z})$ is the composition of $H_2(X,\mathbb{Z}) \to H_1(L,\mathbb{Z})$ with the natural map $H_1(L,\mathbb{Z}) \to H_1(M,\mathbb{Z})$[1)].

Recall that Y is the rational elliptic surface constructed in Section 2, and $\pi\colon Y \to X$ is the contraction of C_1, C_2, E_2, $\ldots$, E_r. Proposition 3.2 gives the short exact sequence

$$0 \to H_2(X^{\mathrm{g}},\mathbb{Z}) \to H_2(X,\mathbb{Z}) \to H_1(M_1,\mathbb{Z}) \oplus H_1(M_2,\mathbb{Z}) \to 0 \tag{3.7}$$

where M_1 (resp. M_2) is the Milnor fiber of the smoothing of $(P_1 \in X)$ (resp. $(P_2 \in X)$). It is well-known that $H_1(M_1,\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H_1(M_2,\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$ (cf. [23, Proposition 13]). Suppose $D \in \operatorname{Pic} Y$ is a divisor such that $(D.C_1) \in 2\mathbb{Z}$, $(D.C_2) \in n\mathbb{Z}$, and $(D.E_2) = \ldots = (D.E_r) = 0$. Then, Theorem 3.1 and (3.7) implies that the cycle $[\pi_* D] \in H_2(X,\mathbb{Z})$ maps to the trivial element of the cokernel $H_1(M_1,\mathbb{Z}) \oplus H_1(M_2,\mathbb{Z})$. In particular, there is a cycle in $H_2(X^{\mathrm{g}})$, which maps to $[\pi_* D]$. Since X^{g} is a nonsingular surface with $p_g = q = 0$, the first Chern class map and Poincaré duality induce an isomorphism $\operatorname{Pic} X^{\mathrm{g}} \simeq H_2(X^{\mathrm{g}},\mathbb{Z})$ (see e.g. [11, §7.1]). We take the line bundle $D^{\mathrm{g}} \in \operatorname{Pic} X^{\mathrm{g}}$ corresponding to $[\pi_* D] \in H_2(X,\mathbb{Z})$. More detailed description of D^{g} will be presented in Proposition 3.6.

The next proposition explains the way to find a preimage along the surjective map $H_2(X,\mathbb{Z}) \to H_1(L_1,\mathbb{Z}) \oplus H_1(L_2,\mathbb{Z})$. Key observation is that if $D \in \operatorname{Pic} Y$, then $[\pi_* D] \in H_2(X,\mathbb{Z})$ maps to

$$\big((D.C_1)\alpha_{C_1},\ (D.C_2)\alpha_{C_2} + (D.E_2)\alpha_{E_2} + \ldots + (D.E_r)\alpha_{E_r}\big).$$

Proposition 3.3. *As in the proof of 2.3, rearrange the chain $C_2, E_2, \ldots, E_r$ as follows:*

$$(G_1,\, G_2,\, \ldots,\, G_r) = (E_{i_k},\, E_{i_{k-1}},\, \ldots,\, E_{i_1},\, C_2,\, E_{j_1},\, \ldots,\, E_{j_\ell}). \tag{3.8}$$

Let α_{G_1}, α_{G_2}, $\ldots$, α_{G_r} be the loops in the plumbing fixture around $G_1 \cup G_2 \cup \ldots \cup G_r$. Assume that α_{C_2} is a generator of the cyclic group $H_1(L_2,\mathbb{Z})$.[2)] *Then there exists a number N' such that $N'\alpha_{C_2} = \alpha_{G_1}$. Now, let N be a solution of the system of congruence equations:*

$$N \equiv \begin{cases} 0 \mod 4 \\ N' \mod n^2. \end{cases}$$

Let $N_1, \ldots, N_9$ be nonnegative integers with $\sum N_i = N$. Then the divisor $D = \lfloor \sum_i N_i \pi^* \pi_* F_i \rfloor$[3)] *on Y has the following properties:*

(a) $(D.G_1) = 1$,

(b) $(D.G_i) = 0$ *for all $i \geq 2$ and* $(D.C_1) = 0$.

---

1)This can be realized as the natural group homomorphism $\mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ (see [11, Lemma 2.1]).

2)This assumption holds if $a = 1$.

3)For a $\mathbb{Q}$-divisor $D = \sum r_i D_i$ with $r_i \in \mathbb{Q}$, $\lfloor D \rfloor$ is defined to be the integral divisor $\sum \lfloor r_i \rfloor D_i$ where $\lfloor - \rfloor$ is the round down function.

Before giving a proof, we need the following lemma.

Lemma 3.4. *Let $k_1, \ldots, k_r \geq 2$ be integers. Then, the system of equations*

$$\begin{bmatrix} k_1 & -1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & k_2 & -1 & \ldots & 0 & 0 & 0 \\ 0 & -1 & k_3 & \ldots & 0 & 0 & 0 \\ & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & k_{r-2} & -1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & k_{r-1} & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & k_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{r-2} \\ x_{r-1} \\ x_r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution in $\{(x_1, \ldots, x_r) \in \mathbb{Q}^r : 0 < x_i < 1,\ \text{for each } i\}$.

Proof. Let $D(k_1, \ldots, k_r)$ be the determinant of the $r \times r$ matrix in the statement. For notational convenience, put $D(\varnothing) = 1$. The solution of this system is given by

$$x_i = \frac{D(k_{i+1}, \ldots, k_r)}{D(k_1, \ldots, k_r)}, \quad i = 1, \ldots, r.$$

For $r \geq 2$, the following identity holds:

$$D(k_1, \ldots, k_r) = k_1 D(k_2, \ldots, k_r) - D(k_3, \ldots, k_r).$$

Using inductive arguments, we find that

$$0 < D(k_r) < D(k_{r-1}, k_r) < \ldots < D(k_2, \ldots, k_r) < D(k_1, \ldots, k_r).$$

In particular, $0 < x_i < 1$ for each $i = 1, \ldots, r$. □

Proof of Proposition 3.3. The divisors $F_1, \ldots, F_9$ are not numerically equivalent, but their intersection with any other type of divisors, namely C_1, G_1, G_2, …, G_r, E_{r+1}, are same. Thus we may assume $D = \lfloor N\pi^*\pi_* F_1 \rfloor$. Factor the map π into the composition $\eta \circ \iota$ where ι is the contraction of G_2, …, G_r and η is the contraction of $(\iota_* C_1)$, $(\iota_* G_1)$. Let X_0 be the target space of the contraction ι. The image of the divisor $N\pi_* F_1$ along $H_2(X, \mathbb{Z}) \to H_1(L_1, \mathbb{Z}) \oplus H_1(L_2, \mathbb{Z})$ is $(0, \alpha_{G_1})$, hence the proper transform D' does not pass through the singular point of X_0. Furthermore, $(D'.\iota_* G_1) = 1$, $(D'.\iota_* C_1) = 0$. It follows that

$$N\eta^*\pi_* F_1 = D' + \frac{1}{-(\iota_* G_1)^2}(\iota_* G_1).$$

Since D' lies on the smooth locus of X_0, $\iota^* D'$ is a Cartier divisor on Y. Now, consider the divisor $\frac{1}{-(\iota_* G_1)^2}\iota^*\iota_* G_1$. There are rational numbers $a_1, \ldots, a_r$ satisfying

$$\frac{1}{-(\iota_* G_1)^2}\iota^*\iota_* G_1 = a_1 G_1 + \ldots + a_r G_r. \tag{3.9}$$

Let $k_i := -G_i^2$. Since $(\iota_* G_1)^2 = (\iota^*\iota_* G_1.G_1)$, taking intersection of (3.9) and G_1 yields the equation $1 = k_1 a_1 - a_2$. Intersections of the equation (3.9) and G_2, …, G_r give rise to the system of linear equations

$$\begin{cases} k_1 a_1 - a_2 = 1 \\ -a_1 + k_2 a_2 - a_3 = 0 \\ \vdots \\ -a_{r-1} + k_r a_r = 0. \end{cases}$$

By Lemma 3.4, $0 < a_1, \dots, a_r < 1$. Consequently, from the equations

$$\begin{aligned} N\pi^*\pi_*F_1 &= \iota^*D' + \frac{1}{-(\iota_*G_1)^2}\iota^*\iota_*G_1 \\ &= \iota^*D' + a_1G_1 + \dots a_rG_r, \end{aligned}$$

we conclude that $D = \lfloor N\pi^*\pi_*F_1 \rfloor = \iota^*D'$. The intersection numbers (1), (2), (3) are easily verified from the above equation. □

Remark 3.5. Proposition 3.3 produces a divisor associated the singular point $P_2 \in X$. Similarly, one can produce a divisor associated to P_1. It suffices to take an integer N such that

$$N \equiv \begin{cases} 1 \bmod 4 \\ 0 \bmod n^2. \end{cases}$$

3.2. **Exceptional vector bundles on X^{g}.** We keep use the notations in Section 2, namely, Y is the rational elliptic surface (Figure 2.2), $\pi\colon Y \to X$ is the contraction in Proposition 2.2. Let $(0 \in T)$ be the base space of the formal versal deformation $\mathcal{X}^{\mathrm{ver}}/(0 \in T)$ of X, and let $(0 \in T_i)$ be the base space of the formal versal deformation $(P_i \in \mathcal{X}^{\mathrm{ver}})/(0 \in T_i)$ of the singularity $(P_i \in X)$. By Lemma 2.4 and [11, Lemma 7.2], there exists a formally smooth morphism of formal schemes

$$\mathfrak{T}\colon (0 \in T) \to \textstyle\prod_i (0 \in T_i).$$

For each $i = 1, 2$, take a base extension $(0 \in T_i') \to (0 \in T_i)$ to which Proposition 6.1 can be applied. Then, there exists a fiber product diagram

$$\begin{tikzcd} (0 \in T') \arrow[r, "\mathfrak{T}'"] \arrow[d] & \prod_i (0 \in T_i') \arrow[d] \\ (0 \in T) \arrow[r, "\mathfrak{T}"] & \prod_i (0 \in T_i) \end{tikzcd} .$$

Let $\mathcal{X}'/(0 \in T')$ be the deformation obtained by pulling back $\mathcal{X}^{\mathrm{ver}}/(0 \in T)$ along $(0 \in T') \to (0 \in T)$. The deformation $\mathcal{X}'/(0 \in T')$ is eligible for Proposition 6.1, hence there exists a proper birational map $\Phi\colon \tilde{\mathcal{X}} \to \mathcal{X}'$ such that the central fiber $\tilde{\mathcal{X}}_0 = \Phi^{-1}(\mathcal{X}_0')$ is the union of three irreducible components $\tilde{X}_0$, W_1, W_2, where $\tilde{X}_0$ is the proper transform of $X = \mathcal{X}_0'$, and W_1 (resp. W_2) is the exceptional locus over P_1 (resp. P_2). The intersection $Z_i := \tilde{X}_0 \cap W_i$ $(i = 1, 2)$ is a smooth rational curve.

From now on, assume $a = 1$. This is the case in which the resolution graph of the singular point $P_2 \in X$ forms the chain $C_2, E_2, \dots, E_r$ in this order. Indeed, the resolution graph of a cyclic quotient singularity $\big(0 \in \mathbb{A}^2/\frac{1}{n^2}(1, n-1)\big)$ is $\underset{-(n+2)}{\bullet} \text{---} \underset{-2}{\bullet} \text{---} \cdots \text{---} \underset{-2}{\bullet}$. Let $\iota\colon Y \to \tilde{X}_0$ be the contraction of $E_2, \dots, E_r$ (see Proposition 6.1(c)). As noted in Remark 6.3, W_1 is isomorphic to $\mathbb{P}^2$, Z_1 is a smooth conic in W_1, hence $\mathcal{O}_{W_1}(1)\big|_{Z_1} = \mathcal{O}_{Z_1}(2)$. Also,

$$W_2 \simeq \mathbb{P}_{x,y,z}(1, n-1, 1), \quad Z_2 = (xy = z^n) \subset W_2, \text{ and } \quad \mathcal{O}_{W_2}(n-1)\big|_{Z_2} = \mathcal{O}_{Z_2}(n). \tag{3.10}$$

The last statement can be verified as follows: let $h_{W_2} = c_1(\mathcal{O}_{W_2}(1))$, then $(n-1)h_{W_2}^2 = 1$, so $\big(c_1(\mathcal{O}_{W_2}(n-1)) \,.\, Z_2\big) = \big((n-1)h_{W_2} \,.\, nh_{W_2}\big) = n$.

In what follows, we construct exceptional vector bundles on the reducible surface $\tilde{\mathcal{X}}_0 = \tilde{X}_0 \cup W_1 \cup W_2$. The following table exhibits the suitable bundles on irreducible components W_1, W_2 with respect to the values $(D.C_1)$, $(D.C_2)$.

$(D.C_1)$	W_1	rank
0	$\mathcal{O}_{W_1}$	1
1	$\mathcal{T}_{W_1}(-1)$	2
2	$\mathcal{O}_{W_1}(2)$	1

$(D.C_2)$	W_2	rank
0	$\mathcal{O}_{W_2}$	1
1	?	n
n	$\mathcal{O}_{W_2}(n-1)$	1

Table 3.1

The symbol $\mathcal{T}_{W_1}$ denotes the tangent sheaf of W_1. The bundle in question mark exists by Proposition 6.2, but we do not use it later. We summarize this observation in the line bundle case: (see Proposition 6.4 to look for the vector bundle case)

Proposition 3.6. *Let $D \in \operatorname{Pic} Y$ be a divisor such that $(D.C_1) = 2d_1 \in 2\mathbb{Z}$, $(D.C_2) = nd_2 \in n\mathbb{Z}$, and $(D.E_i) = 0$ for $i = 2, \dots, r$. Then, there exists a line bundle $\tilde{\mathcal{D}}$ on the reducible surface $\tilde{\mathcal{X}}_0 = \tilde{X}_0 \cup W_1 \cup W_2$ such that*

$$\tilde{\mathcal{D}}\big|_{\tilde{X}_0} = \mathcal{O}_{\tilde{X}_0}(\iota_* D), \quad \tilde{\mathcal{D}}\big|_{W_1} = \mathcal{O}_{W_1}(d_1), \quad \textit{and} \quad \tilde{\mathcal{D}}\big|_{W_2} = \mathcal{O}_{W_2}((n-1)d_2).$$

Using Table 3.1 and Proposition 3.6, we can assemble some exceptional vector bundles on the reducible surface $\tilde{\mathcal{X}}_0 = \tilde{X}_0 \cup W_1 \cup W_2$ (Table 3.2). Due to the exact sequence (3.11), it is not so hard to prove that the bundles listed below are exceptional.

$\tilde{\mathcal{X}}_0$	$\tilde{X}_0$	W_1	W_2
$\mathcal{O}_{\tilde{\mathcal{X}}_0}$	$\mathcal{O}_{\tilde{X}_0}$	$\mathcal{O}_{W_1}$	$\mathcal{O}_{W_2}$
$\tilde{\mathcal{F}}_{ij\ (1\le i\ne j\le 9)}$	$\mathcal{O}_{\tilde{X}_0}(\iota_*(F_i - F_j))$	$\mathcal{O}_{W_1}$	$\mathcal{O}_{W_2}$
$\tilde{\mathcal{C}}_0$	$\mathcal{O}_{\tilde{X}_0}(\iota_* C_0)$	$\mathcal{O}_{W_1}$	$\mathcal{O}_{W_2}$
$\tilde{\mathcal{K}}$	$\mathcal{O}_{\tilde{X}_0}(K_{\tilde{X}_0})$	$\mathcal{O}_{W_1}(1)$	$\mathcal{O}_{W_2}(n-1)$
$\tilde{\mathcal{R}}$	$\mathcal{O}_{\tilde{X}_0}(\iota_* R)^{\oplus 2}$	$\mathcal{T}_{W_1}(-1)$	$\mathcal{O}_{W_2}^{\oplus 2}$

Table 3.2

In the last row, $R = \lfloor N\pi^*\pi_* F_1 \rfloor$ where N is an integer such that

$$N \equiv \begin{cases} 1 \bmod 4 \\ 0 \bmod n^2. \end{cases}$$

See Proposition 3.3 and Remark 3.5.

One of the benefits of having an exceptional vector bundle is that it deforms uniquely to a family.

Definition 3.7. Let $\tilde{\mathcal{D}}$ be an exceptional line bundle on the reducible surface $\tilde{\mathcal{X}}_0$ as in 3.6. Then, $\tilde{\mathcal{D}}_0$ deforms uniquely to a line bundle $\mathscr{D}$ on $\tilde{\mathcal{X}}$. We define $D^{\mathrm{g}} \in \operatorname{Pic} X^{\mathrm{g}}$ by $\mathcal{O}_{X^{\mathrm{g}}}(D^{\mathrm{g}}) = \mathscr{D}\big|_{X^{\mathrm{g}}}$.

We finish this section by presenting an exceptional collection of length 9 on the Dolgachev surface X^{g}. Note that this collection cannot generate the whole category $\mathrm{D^b}(X^{\mathrm{g}})$.

Proposition 3.8. *Let F_{1j}^{g} $(j > 1)$ be the exceptional vector bundle on X^{g}, which arises from the deformation of $\tilde{\mathcal{F}}_{1j}$ along $\tilde{\mathcal{X}}/(0 \in T')$. Then the ordered tuple*

$$\big\langle \mathcal{O}_{X^{\mathrm{g}}}, \mathcal{O}_{X^{\mathrm{g}}}(F_{12}^{\mathrm{g}}), \dots, \mathcal{O}_{X^{\mathrm{g}}}(F_{19}^{\mathrm{g}}) \big\rangle$$

forms an exceptional collection in the derived category $\mathrm{D^b}(X^{\mathrm{g}})$.

Proof. By virtue of upper-semicontinuity, it suffices to prove that $H^p(\tilde{\mathcal{X}}_0, \tilde{\mathcal{F}}_{1i} \otimes \tilde{\mathcal{F}}_{1j}^\vee) = 0$ for $1 \le i < j \le 9$ and $p \ge 0$. For any vector bundle $\tilde{\mathscr{E}}$ on $\tilde{\mathcal{X}}_0$, there is a short exact sequence

$$0 \to \tilde{\mathscr{E}} \to \tilde{\mathscr{E}}\big|_{\tilde{X}_0} \oplus \tilde{\mathscr{E}}\big|_{W_1} \oplus \tilde{\mathscr{E}}\big|_{W_2} \to \tilde{\mathscr{E}}\big|_{Z_1} \oplus \tilde{\mathscr{E}}\big|_{Z_2} \to 0 \tag{3.11}$$

where the morphism at the left is the sum of natural restrictions, and the morphism at right maps (s_0, s_1, s_2) to $(s_0 - s_1, s_0 - s_2)$. It turns out that the above sequence for $\tilde{\mathscr{E}} = \tilde{\mathcal{F}}_{1i} \otimes \tilde{\mathcal{F}}_{1j}^\vee$ becomes

$$0 \to \tilde{\mathcal{F}}_{1i} \otimes \tilde{\mathcal{F}}_{1j}^\vee \to \mathcal{O}_{\tilde{X}_0}(\iota_*(F_j - F_i)) \oplus \mathcal{O}_{W_1} \oplus \mathcal{O}_{W_2} \to \mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_2} \to 0.$$

Since $H^0(\mathcal{O}_{W_k}) \simeq H^0(\mathcal{O}_{Z_k})$ and $H^p(\mathcal{O}_{W_k}) = H^p(\mathcal{O}_{Z_k}) = 0$ for $k = 1, 2$ and $p > 0$, it suffices to prove that $H^p(\mathcal{O}_{\tilde{X}_0}(\iota_*(F_j - F_i))) = 0$ for all $p \ge 0$ and $i < j$. The surface $\tilde{X}_0$ is normal(cf. [11, p. 1178]) and the divisor $F_j - F_i$ does not intersect the exceptional locus of $\iota\colon Y \to \tilde{X}_0$. By Proposition 2.5, $H^p(\tilde{X}_0, \iota_*(F_j - F_i)) \simeq H^p(Y, F_j - F_i)$ for all $p \ge 0$. It remains to prove that $H^p(Y, F_j - F_i) = 0$ for $p \ge 0$. By Riemann-Roch,

$$\chi(F_j - F_i) = \frac{1}{2}(F_j - F_i \,.\, F_j - F_i - K_Y) + 1,$$

and this is zero by Lemma 2.1. Since $(F_j \,.\, F_j - F_i) = -1$ and $F_i \simeq \mathbb{P}^1$, in the short exact sequence

$$0 \to \mathcal{O}_Y(-F_i) \to \mathcal{O}_Y(F_j - F_i) \to \mathcal{O}_{F_i}(F_j) \to 0$$

we obtain $H^0(-F_i) \simeq H^0(F_j - F_i)$. In particular, $H^0(F_j - F_i) = 0$. By Serre duality and Lemma 2.1, $H^2(F_j - F_i) = H^0(E_1 + F_i - F_j - C_2 - \ldots - E_{r+1})^*$. Similarly, since $(E_1 . E_1 + F_i - F_j - C_2 - \ldots - E_{r+1}) < 0$, $(F_i \,.\, F_i - F_i - C_2 - \ldots - E_{r+1}) < 0$ and E_1, F_j are rational curves, $H^0(E_1 + F_i - F_j - C_2 - \ldots - E_{r+1}) \simeq H^0(-F_j - C_2 - \ldots - E_{r+1}) = 0$. This proves that $H^2(F_j - F_i) = 0$. Finally, $\chi(F_j - F_i) = 0$ implies $H^1(F_j - F_i) = 0$. □

Remark 3.9. In Proposition 3.8, the trivial bundle $\mathcal{O}_{X^{\mathrm{g}}}$ can be replaced by the deformation of the line bundle $\tilde{\mathcal{K}}^\vee$ (Table 3.2). The strategy of the proof differs nothing.

Since $\tilde{\mathcal{K}}^\vee$ deforms to $\mathcal{O}_{X^{\mathrm{g}}}(-K_{X^{\mathrm{g}}})$, taking dual shows that

$$\big\langle \mathcal{O}_{X^{\mathrm{g}}}(F_{21}^{\mathrm{g}}),\, \ldots,\, \mathcal{O}_{X^{\mathrm{g}}}(F_{91}^{\mathrm{g}}),\, \mathcal{O}_{X^{\mathrm{g}}}(K_{X^{\mathrm{g}}}) \big\rangle$$

is also an exceptional collection in $\mathrm{D^b}(X^{\mathrm{g}})$. This will be used later (see Step 2 in the proof of Theorem 5.7).

4. The Néron-Severi lattices of Dolgachev surfaces of type $(2, 3)$

This section is devoted to study the simplest case, namely the case $n = 3$ and $a = 1$. The surface Y has simpler configuration (Figure 2.1). We cook up several divisors on X^{g} according to the recipe designed below.

Recipe 4.1. Recall that $\pi\colon Y \to X$ is the contraction of C_1, C_2, E_2 and $\iota\colon Y \to \tilde{X}_0$ is the contraction of E_2.

1. Pick a divisor $D \in \operatorname{Pic} Y$ satisfying $(D.C_1) \in 2\mathbb{Z}$, $(D.C_2) \in 3\mathbb{Z}$, and $(D.E_2) = 0$.
2. Attach suitable line bundles (Proposition 3.6) on W_i $(i = 1, 2)$ to $\mathcal{O}_{\tilde{X}_0}(\iota_* D)$ to produce a line bundle, say $\tilde{\mathcal{D}}$ on $\tilde{\mathcal{X}}_0 = \tilde{X}_0 \cup W_1 \cup W_2$. It deforms to a line bundle $\mathcal{O}_{X^{\mathrm{g}}}(D^{\mathrm{g}})$ on the Dolgachev surface X^{g}.
3. Use the short exact sequence (3.11) to compute $\chi(\tilde{\mathcal{D}})$. Then by deformation invariance of Euler characteristics, $\chi(D^{\mathrm{g}}) = \chi(\tilde{\mathcal{D}})$.

(4) Since the divisor $\pi_* C_0$ is away from the singularities of X, it is Cartier. By Lemma 4.4, $(C_0^{\mathrm{g}}.D^{\mathrm{g}}) = (C_0.D)$. Furthermore, $C_0^{\mathrm{g}} = 6K_{X^{\mathrm{g}}}$, thus the Riemann-Roch formula on the surface X^{g} reads
$$(D^{\mathrm{g}})^2 = \frac{1}{6}(D.C_0) + 2\chi(\tilde{\mathcal{D}}) - 2.$$
This computes the intersections of divisors in X^{g}.

The following lemmas are included for computational purposes.

Lemma 4.2. *Let $h = c_1(\mathcal{O}_{W_2}(1)) \in H_2(W_2, \mathbb{Z})$ be the hyperplane class of the weighted projective space $W_2 = \mathbb{P}(1,2,1)$. For any even integer $n \in \mathbb{Z}$,*
$$\chi(\mathcal{O}_{W_2}(n)) = \frac{1}{4}n(n+4) + 1.$$

Proof. By well-known properties of weighted projective spaces, $(1{\cdot}2{\cdot}1)h^2 = 1$, $c_1(K_{W_2}) = -(1{+}2{+}1)h = -4h$, and $\mathcal{O}_{W_2}(2)$ is invertible. The Riemann-Roch formula for invertible sheaves (cf. [11, Lemma 7.1]) says that $\chi(\mathcal{O}_{W_2}(n)) = \frac{1}{2}(nh\,.\,(n+4)h) + 1 = \frac{1}{4}n(n+4) + 1$. □

Lemma 4.3. *Let S be a projective normal surface with $\chi(\mathcal{O}_S) = 1$. Assume that all the divisors below are supported on the smooth locus of S. Then,*

(a) $\chi(D_1 + D_2) = \chi(D_1) + \chi(D_2) + (D_1.D_2) - 1$;
(b) $\chi(-D) = -\chi(D) + D^2 + 2$;
(c) $\chi(-D) = p_a(D)$ *where $p_a(D)$ is the arithmetic genus of D;*
(d) $\chi(nD) = n\chi(D) + \frac{1}{2}n(n-1)D^2 - n + 1$ *for all $n \in \mathbb{Z}$.*
(d′) $\chi(nD) = n^2\chi(D) + \frac{1}{2}n(n-1)(K_S.D) - n^2 + 1$ *for all $n \in \mathbb{Z}$.*

Assume in addition that D is an integral curve with $p_a(D) = 0$. Then

(e) $\chi(D) = D^2 + 2$, $\chi(-D) = 0$;
(f) $\chi(nD) = \frac{1}{2}n(n+1)D^2 + (n+1)$ *for all $n \in \mathbb{Z}$.*

Proof. All the formula in the statement are simple variants of Riemann-Roch formula. □

Lemma 4.4. *Let D, $\tilde{\mathcal{D}}$, D^{g} as in Recipe 4.1. Then, $(C_0.D) = (C_0^{\mathrm{g}}.D^{\mathrm{g}})$.*

Proof. Since C_0 does not intersect with C_1, C_2, E_2, the corresponding line bundle $\tilde{\mathcal{C}}_0$ on $\tilde{\mathcal{X}}_0$ is the gluing of $\mathcal{O}_{\tilde{X}_0}(\iota_* C_0)$, $\mathcal{O}_{W_1}$, and $\mathcal{O}_{W_2}$. Thus, $(\tilde{\mathcal{D}} \otimes \tilde{\mathcal{C}}_0)\big|_{W_i} = \tilde{\mathcal{D}}\big|_{W_i}$ for $i = 1, 2$. From this and (3.11), it can be immediately shown that $\chi(\tilde{\mathcal{D}} \otimes \tilde{\mathcal{C}}_0) - \chi(\tilde{\mathcal{D}}) = \chi(D + C_0) - \chi(D)$. If D is the trivial divisor on Y, the previous equation tells $\chi(C_0^{\mathrm{g}}) = \chi(\tilde{\mathcal{C}}_0) = \chi(C_0) = 1$. Now, using Lemma 4.3(1), we deduce $(C_0^{\mathrm{g}}.D^{\mathrm{g}}) = \chi(D^{\mathrm{g}} + C_0^{\mathrm{g}}) - \chi(D^{\mathrm{g}}) = \chi(\tilde{\mathcal{D}} \otimes \tilde{\mathcal{C}}_0) - \chi(\tilde{\mathcal{D}}) = \chi(D + C_0) - \chi(D) = (C_0.D)$. □

Definition 4.5. Let $L_0 = p^*(2H)$ be the proper transform of a general plane conic. Then, $(L_0.C_1) = 6$, $(L_0.C_2) = 6$ and $(L_0.E_2) = 0$. Let $\tilde{\mathcal{L}}_0$ be the line bundle on the reducible surface $\tilde{\mathcal{X}}_0 = \tilde{X}_0 \cup W_1 \cup W_2$ such that
$$\tilde{\mathcal{L}}_0\big|_{\tilde{X}_0} = \mathcal{O}_{\tilde{X}_0}(\iota_* L_0), \quad \tilde{\mathcal{L}}_0\big|_{W_1} = \mathcal{O}_{W_1}(3), \text{ and } \quad \tilde{\mathcal{L}}_0\big|_{W_2} = \mathcal{O}_{W_2}(4).$$
This bundle deforms to a line bundle on X^{g}. We denote L_0^{g} its associated Cartier divisor. Let $F_{ij}^{\mathrm{g}} \in \operatorname{Pic} X^{\mathrm{g}}$ be the divisor associated with $\tilde{\mathcal{F}}_{ij}$ (Table 3.2). We define
$$\begin{aligned} G_i^{\mathrm{g}} &:= -L_0^{\mathrm{g}} + 10K_{X^{\mathrm{g}}} + F_{i9}^{\mathrm{g}} \quad \text{for } i = 1, \dots, 8; \\ G_9^{\mathrm{g}} &:= -L_0^{\mathrm{g}} + 11K_{X^{\mathrm{g}}}. \end{aligned}$$

Proposition 4.6. *The following are numerical invariants related to the divisors* $\{G_i^{\mathrm{g}}\}_{1\le i\le 9}$*:*

(a) $\chi(G_i^{\mathrm{g}})=1$ *and* $(G_i^{\mathrm{g}}.K_{X^{\mathrm{g}}})=-1$*;*

(b) *for* $i<j$, $\chi(G_i^{\mathrm{g}}-G_j^{\mathrm{g}})=0$*.*

Furthermore, $(G_i^{\mathrm{g}})^2=-1$ *and* $(G_i^{\mathrm{g}}.G_j^{\mathrm{g}})=0$ *for* $1\le i<j\le 9$*.*

Proof. First, consider the case $i\le 8$. By Recipe 4.1(4) and $K_{X^{\mathrm{g}}}^2=0$, $(K_{X^{\mathrm{g}}}.G_i^{\mathrm{g}})=\frac{1}{6}(C_0.-L_0+F_i-F_9)=-1$. Since the alternating sum of Euler characteristics in the sequence (3.11) is zero, we get the formula

$$\begin{aligned}\chi(\tilde{\mathcal{L}}_0^\vee\otimes\tilde{\mathcal{F}}_{i9})&=\chi(-L_0+F_i-F_9)+\chi(\mathcal{O}_{W_1}(-3))+\chi(\mathcal{O}_{W_2}(-4))\\&\quad-\chi(\mathcal{O}_{Z_1}(-6))-\chi(\mathcal{O}_{Z_2}(-6)),\end{aligned}$$

which computes $\chi(\tilde{\mathcal{L}}_0^\vee\otimes\tilde{\mathcal{F}}_{i9})=11$. The Riemann-Roch formula for $-L_0^{\mathrm{g}}+F_{i9}^{\mathrm{g}}=G_i^{\mathrm{g}}-10K_{X^{\mathrm{g}}}$ says $(G_i^{\mathrm{g}}-10K_{X^{\mathrm{g}}})^2-(K_{X^{\mathrm{g}}}\,.\,G_i^{\mathrm{g}}-K_{X^{\mathrm{g}}})=20$, hence $(G_i^{\mathrm{g}})^2=-1$. Using Riemann-Roch again, we derive $\chi(G_i^{\mathrm{g}})=1$. For $1\le i<j\le 8$, $G_i-G_j=F_i-F_j$. Since $(F_i-F_j\,.C_1)=(F_i-F_j\,.C_2)=(F_i-F_j\,.E_2)=0$, the line bundle $\mathcal{O}_X(\pi_*F_i-\pi_*F_j)$ deforms to the Cartier divisor F_{ij}^{g}. Hence, $\chi(G_i^{\mathrm{g}}-G_j^{\mathrm{g}})=\chi(F_i-F_j)=0$. This proves the statement for $i,j\le 8$. The proof of the statement involving G_9^{g} follows the same lines. Since $\chi(\tilde{\mathcal{L}}_0^\vee)=12$, $(G_9^{\mathrm{g}}-11K_{X^{\mathrm{g}}})^2-(K_{X^{\mathrm{g}}}\,.\,G_9^{\mathrm{g}}-11K_{X^{\mathrm{g}}})=22$. This leads to $(G_9^{\mathrm{g}})^2=-1$. For $i\le 8$,

$$\begin{aligned}\chi(G_i^{\mathrm{g}}-G_9^{\mathrm{g}})&=\chi(F_i-F_9-K_Y)+\chi(\mathcal{O}_{W_1}(-1))+\chi(\mathcal{O}_{W_2}(-2))\\&\quad-\chi(\mathcal{O}_{Z_1}(-2))-\chi(\mathcal{O}_{Z_2}(-3)),\end{aligned}$$

and the right hand side is zero. □

We complete our list of divisors in $\operatorname{Pic}X^{\mathrm{g}}$ by introducing G_{10}^{g}. The choice of G_{10}^{g} is motivated by the proof of the step (iii) ⇒ (i) in [29, Theorem 2.1].

Proposition 4.7. *Let* G_{10}^{g} *be the* $\mathbb{Q}$*-divisor* $\frac{1}{3}(G_1^{\mathrm{g}}+G_2^{\mathrm{g}}+\ldots+G_9^{\mathrm{g}}-K_{X^{\mathrm{g}}})$*. Then,* G_{10}^{g} *is a Cartier divisor.*

Proof. Since

$$\sum_{i=1}^{9}G_i^{\mathrm{g}}-K_{X^{\mathrm{g}}}=-9L_0^{\mathrm{g}}+90K_{X^{\mathrm{g}}}+\sum_{i=1}^{8}F_{i9}^{\mathrm{g}},$$

it suffices to prove that $\sum\limits_{i=1}^{8}F_{i9}^{\mathrm{g}}=3D^{\mathrm{g}}$ for some $D^{\mathrm{g}}\in\operatorname{Pic}X^{\mathrm{g}}$. Let $p\colon Y\to\mathbb{P}^2$ be the blowing up morphism and let H be a line in $\mathbb{P}^2$. Since $K_Y=p^*(-3H)+F_1+F_2+\ldots+F_9+E_1+E_2+2E_3$, $K_Y-E_1-E_2-2E_3=p^*(-3H)+F_1+\ldots+F_9=-C_0$, so $F_1+\ldots+F_9=3p^*H-C_0$. Consider the divisor p^*H-3F_9 in Y. Clearly, the intersections of (p^*H-3F_9) with C_1, C_2, E_2 are all zero, hence $\pi_*(p^*H-3F_9)$ deforms to a Cartier divisor $(p^*H-3F_9)^{\mathrm{g}}$ in X^{g}. Since

$$\begin{aligned}\sum_{i=1}^{8}(F_i-F_9)&=\sum_{i=1}^{9}F_i-9F_9\\&=3(p^*H-3F_9)-C_0\end{aligned}$$

and C_0^{g} deforms to $6K_{X^{\mathrm{g}}}$, $D^{\mathrm{g}}:=(p^*H-3F_9)^{\mathrm{g}}-2K_{X^{\mathrm{g}}}$ satisfies $\sum_{i=1}^{8}F_{i9}^{\mathrm{g}}=3D^{\mathrm{g}}$. □

Combining the propositions 4.6 and 4.7, we obtain:

Theorem 4.8. *The intersection matrix of divisors* $\{G_i^{\mathrm{g}}\}_{i=1}^{10}$ *is*

$$\Big((G_i^{\mathrm{g}}.G_j^{\mathrm{g}})\Big)_{1\le i,j\le 10} = \begin{bmatrix} -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{4.12}$$

In particular, the set $G := \{G_i^{\mathrm{g}}\}_{i=1}^{10}$ *forms a* $\mathbb{Z}$*-basis of the Néron-Severi lattice* $\mathrm{NS}(X^{\mathrm{g}})$*. By* [7, p. 137], $\mathrm{Pic}\, X^{\mathrm{g}}$ *is torsion-free, thus it describes* $\mathrm{Pic}\, X^{\mathrm{g}}$ *completely.*

Proof. We claim that the divisors $\{G_i^{\mathrm{g}}\}_{i=1}^{10}$ generate the Néron-Severi lattice. By Hodge index theorem, there is a $\mathbb{Z}$-basis for $\mathrm{NS}(X^{\mathrm{g}})$, say $\alpha = \{\alpha_i\}_{i=1}^{10}$, such that the intersection matrix with respect to $\{\alpha_i\}_{i=1}^{10}$ is same as (4.12). Let $A = (a_{ij})_{1\le i,j\le 10}$ be the integral matrix defined by

$$G_i^{\mathrm{g}} = \sum_{j=1}^{10} a_{ij}\alpha_j.$$

Given $v \in \mathrm{NS}(X^{\mathrm{g}})$, let $[v]_G$ (resp. $[v]_\alpha$) be the column matrix of coordinates with respect to the basis G (resp. α). Then, $[v]_\alpha = A[v]_G$. For $v_1, v_2 \in \mathrm{NS}(X^{\mathrm{g}})$,

$$\begin{aligned}(v_1.v_2) &= [v_1]_\alpha^{\mathrm{t}} E [v_1]_\alpha \\ &= [v_1]_G^{\mathrm{t}} A^{\mathrm{t}} E A [v_1]_G,\end{aligned}$$

where E is the intersection matrix with respect to the basis α. The above equation implies that the intersection matrix with respect to G is $A^{\mathrm{t}}EA$. Since the intersection matrices with respect to both bases are same, $E = A^{\mathrm{t}}EA$. This implies that $1 = \det(A^{\mathrm{t}}A) = (\det A)^2$, hence A is invertible over $\mathbb{Z}$. This proves that G is a $\mathbb{Z}$-basis of $\mathrm{NS}(X^{\mathrm{g}})$. The last statement on the Picard group follows immediately. ☐

We close this section with the summary of divisors on X^{g}.

Summary 4.9. Recall that Y is the rational elliptic surface in Section 2, $p\colon Y \to \mathbb{P}^2$ is the morphism of blowing up, $H \in \mathrm{Pic}\,\mathbb{P}^2$ is a hyperplane divisor, and $\pi\colon Y \to X$ is the contraction of C_1, C_2, E_2. Then,

(1) F_{ij}^{g} $(1 \le i,j \le 9)$ is the divisor associated with $F_i - F_j$;
(2) $(p^*H - 3F_9)^{\mathrm{g}}$ is the divisor obtained from $p^*H - 3F_9$;
(3) L_0^{g} is the divisor induced by the proper transform of a general conic $p^*(2H)$;
(4) $G_i^{\mathrm{g}} = -L_0^{\mathrm{g}} + 10K_{X^{\mathrm{g}}} + F_{i9}^{\mathrm{g}}$ for $i = 1, \dots, 8$;
(5) $G_9^{\mathrm{g}} = -L_0^{\mathrm{g}} + 11K_{X^{\mathrm{g}}}$;
(6) $G_{10}^{\mathrm{g}} = -3L_0^{\mathrm{g}} + (p^*H - 3F_9)^{\mathrm{g}} + 28K_{X^{\mathrm{g}}}$.

5. Exceptional collections of maximal length on Dolgachev surfaces of type (2, 3)

5.1. **Exceptional collection of maximal length.** We continue to study the case $n = 3$ and $a = 1$. Throughout this section, we will prove that there exists an exceptional collection of maximal length in $\mathrm{D^b}(X^{\mathrm{g}})$ for a cubic pencil $|\lambda p_*C_1 + \mu p_*C_2|$ generated by two general plane nodal cubics. Proving exceptionality of a given collection usually consists of numerous cohomology computations, so we begin with some computational machineries.

Lemma 5.1. *The line bundle, which is the glueing of $\mathcal{O}_{\tilde{X}_0}(\iota_* C_1)$, $\mathcal{O}_{W_1}(-2)$ and $\mathcal{O}_{W_2}$, deforms to the trivial line bundle on X^{g}. Similarly, the glueing of $\mathcal{O}_{\tilde{X}_0}(\iota_*(2C_2+E_2))$, $\mathcal{O}_{W_1}$, and $\mathcal{O}_{W_2}(-6)$ deforms to the trivial line bundle on X^{g}.*

Proof. Let $\tilde{\mathcal{C}}_1$ be the glueing of line bundles $\mathcal{O}_{\tilde{X}_0}(\iota_* C_1)$, $\mathcal{O}_{W_1}(-2)$, and $\mathcal{O}_{W_2}$, and let $\mathcal{O}_{X^{\mathrm{g}}}(C_1^{\mathrm{g}})$ be the deformed line bundle. It is immediate to see that $\chi(C_1^{\mathrm{g}}) = 1$ and $\chi(-C_1^{\mathrm{g}}) = 1$. By Riemann-Roch formula, $(C_1^{\mathrm{g}})^2 = (C_1^{\mathrm{g}}.K_{X^{\mathrm{g}}}) = 0$. For $i \le 8$,

$$\begin{aligned}\chi(G_i^{\mathrm{g}} - 10K_{X^{\mathrm{g}}} - C_1^{\mathrm{g}}) &= \chi(\tilde{\mathcal{L}}_0^\vee \otimes \tilde{\mathcal{F}}_{i9} \otimes \tilde{\mathcal{C}}_1^\vee) \\ &= \chi(-L_0 + F_i - F_9 - C_1) + \chi(\mathcal{O}_{W_1}(-1)) + \chi(\mathcal{O}_{W_2}(-4)) \\ &\quad - \chi(\mathcal{O}_{Z_1}(-2)) - \chi(\mathcal{O}_{Z_2}(-6)).\end{aligned}$$

This computes $\chi(G_i^{\mathrm{g}} - 10K_{X^{\mathrm{g}}} - C_1^{\mathrm{g}}) = 11$. By Riemann-Roch, $(G_i^{\mathrm{g}} - 10K_{X^{\mathrm{g}}} - C_1^{\mathrm{g}})^2 - (K_{X^{\mathrm{g}}} \,.\, G_i^{\mathrm{g}} - 10K_{X^{\mathrm{g}}} - C_1^{\mathrm{g}}) = 2\chi(G_i^{\mathrm{g}} - 10K_{X^{\mathrm{g}}} - C_1^{\mathrm{g}}) - 2 = 20$. The left hand side is $-2(G_i^{\mathrm{g}}.C_1^{\mathrm{g}}) + 20$, thus $(G_i^{\mathrm{g}}.C_1^{\mathrm{g}}) = 0$. Since $(C_1^{\mathrm{g}}.K_{X^{\mathrm{g}}}) = 0$ and $3G_{10}^{\mathrm{g}} = G_1^{\mathrm{g}} + \ldots + G_9^{\mathrm{g}} - K_{X^{\mathrm{g}}}$, $(G_{10}^{\mathrm{g}}.C_1^{\mathrm{g}}) = 0$. Hence, C_1^{g} is numerically trivial by Theorem 4.8. This shows that C_1^{g} is trivial since there is no torsion in $\operatorname{Pic} X^{\mathrm{g}}$. Exactly the same proof is valid for the line bundle coming from $2C_2 + E_2$. □

Definition 5.2. Let $D \in \operatorname{Pic} Y$ be a divisor such that $(D.C_1) \in 2\mathbb{Z}$, $(D.C_2) \in 3\mathbb{Z}$, and $(D.E_2) = 0$. Then, Recipe 4.1 produces a Cartier divisor $D^{\mathrm{g}} \in \operatorname{Pic} X^{\mathrm{g}}$ from D. In this case, we say D *deforms to* D^{g}. This is a slight abuse of terminology; it is not D, but $\iota_* D$ that deforms to D^{g}.

Example 5.3. Since C_0 deforms to $6K_{X^{\mathrm{g}}}$, $2E_1 = C_0 - C_1$ deforms to $6K_{X^{\mathrm{g}}}$. Thus E_1 deforms to $3K_{X^{\mathrm{g}}}$. Similarly, $C_2 + E_2 + E_3$ deforms to $2K_{X^{\mathrm{g}}}$. Hence, $K_Y = E_1 - C_2 - E_2 - E_3$ deforms to $3K_{X^{\mathrm{g}}} - 2K_{X^{\mathrm{g}}} = K_{X^{\mathrm{g}}}$. Also, $(E_2 + 2E_3) - E_1$ deforms to $K_{X^{\mathrm{g}}}$, whereas K_Y and $(E_2 + 2E_3) - E_1$ are different in $\operatorname{Pic} Y$. These are in principle due to Lemma 5.1. For instance, we have

$$\begin{aligned}(E_2 + 2E_3) - E_1 - K_Y &= -2E_1 + C_2 + 2E_2 + 3E_3 \\ &= -C_1,\end{aligned}$$

thus $(E_2 + 2E_3)^{\mathrm{g}} - E_1^{\mathrm{g}} - K_{X^{\mathrm{g}}} = -C_1^{\mathrm{g}} = 0$.

As Example 5.3 presents, we can take various $D \in \operatorname{Pic} Y$, which deforms to a fixed divisor $D^{\mathrm{g}} \in \operatorname{Pic} X^{\mathrm{g}}$. The following lemma gives a direction to choose D. Note that the lemma requires some conditions on $(D.C_1)$ and $(D.C_2)$, but Lemma 5.1 provides the way to adjust them.

Lemma 5.4. *Let D be a divisor in Y such that $(D.C_1) = 2d_1 \in 2\mathbb{Z}$, $(D.C_2) = 3d_2 \in 3\mathbb{Z}$, and $(D.E_2) = 0$. Let D^{g} be the deformation of D. Then,*

$$h^0(X^{\mathrm{g}}, D^{\mathrm{g}}) \le h^0(Y, D) + h^0(\mathcal{O}_{W_1}(d_1)) + h^0(\mathcal{O}_{W_2}(2d_2)) - h^0(\mathcal{O}_{Z_1}(2d_1)) - h^0(\mathcal{O}_{Z_2}(3d_3)).$$

In particular, if $d_1, d_2 \le 1$, then $h^0(X^{\mathrm{g}}, D^{\mathrm{g}}) \le h^0(Y, D)$.

Proof. Since $(D.E_2) = 0$, we have $H^p(\tilde{X}_0, \iota_* D) \simeq H^p(Y, D)$ for all $p \ge 0$ (Proposition 2.5). Recall that there exists a short exact sequence (introduced in (3.11))

$$0 \to \tilde{\mathcal{D}} \to \mathcal{O}_{\tilde{X}_0}(\iota_* D) \oplus \mathcal{O}_{W_1}(d_1) \oplus \mathcal{O}_{W_2}(2d_2) \to \mathcal{O}_{Z_1}(2d_1) \oplus \mathcal{O}_{Z_2}(3d_2) \to 0, \tag{5.13}$$

where $\tilde{\mathcal{D}}$ is the line bundle constructed as in Proposition 3.6, and the notations W_i, Z_i are explained in (3.10). We first claim the following: if $d_1, d_2 \le 1$, then the maps $H^0(\mathcal{O}_{W_1}(d_1)) \to H^0(\mathcal{O}_{Z_1}(2d_1))$ and

$H^0(\mathcal{O}_{W_2}(2d_2)) \to H^0(\mathcal{O}_{Z_2}(3d_2))$ are isomorphisms. Only the nontrivial cases are $d_1 = 1$ and $d_2 = 1$. Since Z_1 is a smooth conic in $W_1 = \mathbb{P}^2$, there is a short exact sequence

$$0 \to \mathcal{O}_{W_1}(-1) \to \mathcal{O}_{W_1}(1) \to \mathcal{O}_{Z_1}(2) \to 0.$$

All the cohomology groups of $\mathcal{O}_{W_1}(-1)$ vanish, so $H^p(\mathcal{O}_{W_1}(1)) \simeq H^p(\mathcal{O}_{Z_1}(2))$ for all $p \geq 0$. In the case $d_2 = 1$, we consider

$$0 \to \mathcal{I}_{Z_2}(2) \to \mathcal{O}_{W_2}(2) \to \mathcal{O}_{Z_2}(3) \to 0,$$

where $\mathcal{I}_{Z_2} \subset \mathcal{O}_{W_2}$ is the ideal sheaf of the closed subscheme $Z_2 = (xy = z^3) \subset \mathbb{P}_{x,y,z}(1,2,1)$. The ideal $(xy - z^3)$ does not contain any nonzero homogeneous element of degree 2, so $H^0(\mathcal{I}_{Z_2}(2)) = 0$. This shows that $H^0(\mathcal{O}_{W_2}(2)) \to H^0(\mathcal{O}_{Z_2}(3))$ is injective. Furthermore, $H^0(\mathcal{O}_{W_2}(2))$ is generated by x^2, xz, z^2, y, hence $h^0(\mathcal{O}_{W_2}(2)) = h^0(\mathcal{O}_{Z_3}(3)) = 4$. This proves that $H^0(\mathcal{O}_{W_2}(2)) \simeq H^0(\mathcal{O}_{Z_3}(3))$, as desired. If $d_1, d_2 > 1$, it is clear that $H^0(\mathcal{O}_{W_1}(d_1)) \to H^0(\mathcal{O}_{Z_1}(2d_1))$ and $H^0(\mathcal{O}_{W_2}(2d_2)) \to H^0(\mathcal{O}_{Z_2}(3d_2))$ are surjective.

The cohomology long exact sequence of (5.13) begins with

$$\begin{aligned} 0 \to H^0(\tilde{\mathcal{D}}) \to H^0(\iota_* D) \oplus H^0(\mathcal{O}_{W_1}(d_1)) \oplus H^0(\mathcal{O}_{W_2}(2d_2)) \\ \to H^0(\mathcal{O}_{Z_1}(2d_1)) \oplus H^0(\mathcal{O}_{Z_2}(3d_2)). \end{aligned}$$

By the previous arguments, the last map is surjective. Indeed, the image of $(0, s_1, s_2) \in H^0(\iota_* D) \oplus H^0(\mathcal{O}_{W_1}(d_1)) \oplus H^0(\mathcal{O}_{W_2}(2d_2))$ is $(-s_1\big|_{Z_1}, -s_2\big|_{Z_2})$. The upper-semicontinuity of cohomologies establishes the inequality in the statement. $\square$

The next lemma is useful to remove redundant parts of D in H^0 computations.

Lemma 5.5. *Let S be a nonsingular projective surface, and let D be a divisor on S. For a nonsingular projective curve C in S, suppose $(D.C) < 0$.*

(a) *If $C^2 \geq 0$, then $H^0(D) = 0$.*

(b) *If $C^2 < 0$, then $H^0(D) \simeq H^0(D - mC)$ for all $0 < m \leq \lceil \frac{(D.C)}{(C.C)} \rceil$.*

Proof. In the short exact sequence

$$0 \to \mathcal{O}_S(D - C) \to \mathcal{O}_S(D) \to \mathcal{O}_C(D) \to 0,$$

$H^0(\mathcal{O}_C(D)) = 0$, thus $H^0(D) \simeq H^0(D - C)$. If $C^2 \geq 0$ and $m > 0$, then $(D - mC.C) = (D.C) - mC^2 < 0$, so $H^0(D - (m+1)C) \simeq H^0(D - mC)$. For an ample divisor A, $(D - mC\,.\,A) < 0$ for $m \gg 0$, hence $D - mC$ cannot be effective. This proves (a). If $C^2 < 0$, let m_0 be the largest number satisfying $(D - (m_0 - 1)C\,.\,C) < 0$. Then, $H^0(D - m_0 C) \simeq H^0(D)$ by the previous argument. Since

$$(D - mC\,.\,C) \geq 0 \Leftrightarrow m \geq \frac{(D.C)}{(C.C)},$$

m_0 is the smallest integer greater than or equal to $\frac{(D.C)}{(C.C)}$, thus $m_0 = \lceil \frac{(D.C)}{(C.C)} \rceil$. $\square$

By [29, Theorem 3.1], it can be shown that the collection (5.14) in the theorem below is a numerically exceptional collection. Our aim is to prove that (5.14) is indeed an exceptional collection in $\mathrm{D^b}(X^{\mathrm{g}})$. Before proceed to the theorem, we introduce one terminology.

Definition 5.6. During the construction of Y, the node of p_*C_2 is blown up twice. The second blow up at p_*C_2 corresponds to one of the two tangent directions at the node of p_*C_2. We refer to the tangent direction corresponding to the second blow up as the *distinguished tangent* direction at the node of p_*C_2.

Theorem 5.7. *Suppose X^{g} is originated from a cubic pencil $|\lambda p_*C_1 + \mu p_*C_2|$ which is generated by two general plane nodal cubics. Let $G_1^{\mathrm{g}}, \ldots, G_{10}^{\mathrm{g}}$ as in 4.9, let G_0^{g} be the trivial divisor, and let $G_{11}^{\mathrm{g}} = 2G_{10}^{\mathrm{g}}$. For notational simplicity, we denote the rank of $\mathrm{Ext}^p(G_i^{\mathrm{g}}, G_j^{\mathrm{g}})(= H^p(-G_i^{\mathrm{g}} + G_j^{\mathrm{g}}))$ by h_{ij}^p. The following table describes $\mathbb{R}\mathrm{Hom}(G_i^{\mathrm{g}}, G_j^{\mathrm{g}})$. For example, the triple of (G_9^{g}-row, G_{10}^{g}-column), which is $(0\,0\,2)$, means that $(h_{9,10}^0,\ h_{9,10}^1,\ h_{9,10}^2) = (0, 0, 2)$.*

	G_0^{g}	$G_{1\le i\le 8}^{\mathrm{g}}$	G_9^{g}	G_{10}^{g}	G_{11}^{g}
G_0^{g}	1 0 0	0 0 1	0 0 1	0 0 3	0 0 6
$G_{1\le i\le 8}^{\mathrm{g}}$		1 0 0		0 0 2	0 0 5
G_9^{g}			1 0 0	0 0 2	0 0 5
G_{10}^{g}				1 0 0	0 0 3
G_{11}^{g}					1 0 0

Table 5.3

The blanks stand for $0\,0\,0$, and $h_{ij}^p = 0$ for all p and $1 \le i \ne j \le 8$. In parcicular, the collection

$$\big\langle \mathcal{O}_{X^{\mathrm{g}}}(G_0^{\mathrm{g}}),\ \mathcal{O}_{X^{\mathrm{g}}}(G_1^{\mathrm{g}}),\ \ldots,\ \mathcal{O}_{X^{\mathrm{g}}}(G_{10}^{\mathrm{g}}),\ \mathcal{O}_{X^{\mathrm{g}}}(G_{11}^{\mathrm{g}})\big\rangle \tag{5.14}$$

is an exceptional collection of length 12 in $\mathrm{D^b}(X^{\mathrm{g}})$.

Proof. Recall that (see Summary 4.9)

$$\begin{aligned} G_i^{\mathrm{g}} &= -L_0^{\mathrm{g}} + F_{i9}^{\mathrm{g}} + 10K_{X^{\mathrm{g}}},\ \ i = 1, \ldots, 8; \\ G_9^{\mathrm{g}} &= -L_0^{\mathrm{g}} + 11K_{X^{\mathrm{g}}}; \\ G_{10}^{\mathrm{g}} &= -3L_0^{\mathrm{g}} + (p^*H - 3F_9)^{\mathrm{g}} + 28K_{X^{\mathrm{g}}}; \\ G_{11}^{\mathrm{g}} &= -6L_0^{\mathrm{g}} + 2(p^*H - 3F_9)^{\mathrm{g}} + 56K_{X^{\mathrm{g}}}. \end{aligned}$$

The proof consists of numerous cohomology vanishings for which we divide into several steps. The numerical computations are collected in Dictionary 5.12. Note that we can always evaluate $\chi(-G_i^{\mathrm{g}} + G_j^{\mathrm{g}}) = \sum_p (-1)^p h_{ij}^p$, thus it suffices to compute only two (mostly h^0 and h^2) of $\{h_{ij}^p : p = 0, 1, 2\}$.

In the first part of the proof, we deduce the following table using numerical methods.

	G_0^{g}	$G_{1\le i\le 8}^{\mathrm{g}}$	G_9^{g}	G_{10}^{g}	G_{11}^{g}
G_0^{g}	1 0 0	0 0 1	0 0 1	χ=3	χ=6
$G_{1\le i\le 8}^{\mathrm{g}}$	0 0 0	1 0 0	0 0 0	0 0 2	χ=5
G_9^{g}	χ=0	0 0 0	1 0 0	0 0 2	χ=5
G_{10}^{g}	χ=0	χ=0	χ=0	1 0 0	χ=3
G_{11}^{g}	χ=0	χ=0	χ=0	χ=0	1 0 0

Table 5.4

The slots with $\chi = d$ means $\chi(-G_i^{\mathrm{g}} + G_j^{\mathrm{g}}) = \sum_p (-1)^p h_{ij}^p = d$. For those slots, we do not compute each h_{ij}^p for the moment. In the end, they will be completed through a different approach.

Step 1. As explained above, the collection (5.14) is numerically exceptional, hence $\chi(-G_i^{\mathrm{g}} + G_j^{\mathrm{g}}) = \sum_p h_{ij}^p = 0$ for all $0 \le j < i \le 11$. Furthermore, the surface X^{g} is minimal, thus $K_{X^{\mathrm{g}}}$ is nef. It follows that $h^0(D^{\mathrm{g}}) = 0$ if D^{g} is $K_{X^{\mathrm{g}}}$-negative, and $h^2(D^{\mathrm{g}}) = 0$ if D^{g} is $K_{X^{\mathrm{g}}}$-positive. Since

$$(K_{X^{\mathrm{g}}}.G_i^{\mathrm{g}}) = \begin{cases} -1 & i \le 9 \\ -3 & i = 10 \\ -6 & i = 11, \end{cases}$$

this already enforces a number of cohomologies to be zero. Indeed, all the numbers in the following list are zero:

$$\{h_{0i}^0\}_{i\le 11},\ \{h_{i,10}^0, h_{i,11}^0\}_{i\le 9},\ \{h_{i0}^2\}_{i\le 11},\ \{h_{10,i}^2, h_{11,i}^2\}_{i\le 9}.$$

Step 2. If $1 \le j \ne i \le 8$, then $-G_i^{\mathrm{g}} + G_j^{\mathrm{g}}$ can be realized as $-F_i + F_j$ in the rational elliptic surface Y. Hence,

$$\big\langle \mathcal{O}_{X^{\mathrm{g}}}(G_1^{\mathrm{g}}),\ \ldots,\ \mathcal{O}_{X^{\mathrm{g}}}(G_8^{\mathrm{g}}) \big\rangle$$

is an exceptional collection by Proposition 3.8. This proves that $h_{ij}^p = 0$ for all $p \ge 0$ and $1 \le i \ne j \le 8$. Also, $-G_9^{\mathrm{g}} + G_i^{\mathrm{g}} = -K_{X^{\mathrm{g}}} + F_{i9}^{\mathrm{g}}$ for $1 \le i \le 8$. Remark 3.9 shows that $h_{9i}^p = h^p(-K_{X^{\mathrm{g}}} + F_{i9}^{\mathrm{g}}) = 0$ for $p \ge 0$ and $1 \le i \le 8$. Furthermore, by Serre duality, $h_{i9}^p = h^{2-p}(F_{i9}^{\mathrm{g}}) = 0$ for all $p \ge 0$ and $1 \le i \le 8$.

Step 3. We verify Table 5.4 using the following strategy:

(1) If we want to compute h_{ij}^0, then pick $D_{ij}^{\mathrm{g}} := -G_i^{\mathrm{g}} + G_j^{\mathrm{g}}$. If the aim is to evaluate h_{ij}^2, then take $D_{ij}^{\mathrm{g}} := K_{X^{\mathrm{g}}} + G_i^{\mathrm{g}} - G_j^{\mathrm{g}}$, so that $h_{ij}^2 = h^0(D_{ij}^{\mathrm{g}})$ by Serre duality.

(2) Express D_{ij}^{g} in terms of L_0^{g}, $(p^*H - 3F_9)^{\mathrm{g}}$, F_{i9}^{g}, and $K_{X^{\mathrm{g}}}$. Via Summary 4.9, we can translate L_0^{g}, $(p^*H - 3F_9)^{\mathrm{g}}$, F_{i9}^{g} into the divisors on Y. Further, we have $6K_{X^{\mathrm{g}}} = C_0^{\mathrm{g}}$, $3K_{X^{\mathrm{g}}} = E_1^{\mathrm{g}}$, and $2K_{X^{\mathrm{g}}} = (C_2 + E_2 + E_3)^{\mathrm{g}}$, thus an arbitrary integer multiple of $K_{X^{\mathrm{g}}}$ also can be translated into divisors on Y. Together with these translations, use Lemma 5.1 to find a Cartier divisor D_{ij} on Y, which deforms to D_{ij}^{g}, and satisfies $(D_{ij}.C_1) \le 2$, $(D_{ij}.C_2) \le 3$, $(D_{ij}.E_2) = 0$.

(3) Compute an upper bound of $h^0(D_{ij})$. Then by Lemma 5.4, $h^0(D_{ij}^{\mathrm{g}}) \le h^0(D_{ij})$.

(4) In any occassions, we will find that the upper bound obtained in (3) coincides with $\chi(-G_i^{\mathrm{g}} + G_j^{\mathrm{g}})$. Also, at least one of $\{h_{ij}^0, h_{ij}^2\}$ is zero by Step 1. From this we deduce $h^0(D_{ij}^{\mathrm{g}}) \ge$ (the upper bound obtained in (3)), hence the equality holds. Consequently, the numbers $\{h_{ij}^p : p = 0, 1, 2\}$ are evaluated.

Step 4. We follow the strategy in Step 3 to complete Table 5.4. Let $i \in \{1, \ldots, 8\}$. To verify $h_{i0}^0 = 0$, we take $D_{i0}^{\mathrm{g}} = -G_i^{\mathrm{g}} = L_0^{\mathrm{g}} - F_{i9}^{\mathrm{g}} - 10K_{X^{\mathrm{g}}}$. Translation into the divisors on Y gives:

$$D'_{i0} = p^*(2H) + F_9 - F_i - 2C_0 + (C_2 + E_2 + E_3)$$

Since $(D'_{i0}.C_1) = 6$ and $(D'_{i0}.C_2) = 3$, we replace the divisor D'_{i0} by $D_{i0} := D'_{i0} + C_1$ so that the condition $(D_{i0}.C_1) \le 2$ is fulfilled. Now, $h^0(D_{i0}) = 0$ by Dictionary 5.12(1), thus $h_{i0}^0 \le h^0(D_{i0}) = 0$ by Lemma 5.4. Finally, $\chi(-G_i^{\mathrm{g}}) = 0$ and $h_{i0}^2 = 0$ (Step 1), hence $h_{i0}^1 = 0$.

We repeat this routine to the following divisors:

$$\begin{aligned}
D_{0i} &= p^*(2H) + F_9 - F_i - C_0 + C_1 - E_1 + (2C_2 + E_2); \\
D_{09} &= p^*(2H) - 2C_0 + C_1 + (C_2 + E_2 + E_3); \\
D_{i,10} &= p^*(3H) + 2F_9 + F_i - 2C_0 + 2C_1 - E_1 - (C_2 + E_2 + E_3) + 2(2C_2 + E_2); \\
D_{9,10} &= p^*(3H) + 3F_9 - 3C_0 + 3C_1 + (C_2 + E_2 + E_3) + (2C_2 + E_2).
\end{aligned}$$

Together with Dictionary 5.12(2–5), all the slots of Table 5.4 are verified.

Step 5. It is difficult to complete Table 5.3 using the numerical argument (see for example, Remark 5.8). We introduce another plan to overcome these difficulties.

(1) Take $D^{\mathrm{g}}_{ij} \in \operatorname{Pic} X^{\mathrm{g}}$ and $D_{ij} \in \operatorname{Pic} Y$ as in Step 3(1–2). We may assume $(D_{ij}.C_1) \in \{0, 2\}$ and $(D_{ij}.C_2) \in \{-3, 0, 3\}$. If $(D_{ij}.C_2) = -3$, $h^0(D_{ij}) = h^0(D_{ij} - C_2 - E_2)$, thus it suffices to prove that $H^0(D_{ij} - C_2 - E_2) = 0$. Hence, we replace D_{ij} by $D_{ij} - C_2 - E_2$ if $(D.C_2) = -3$. In some occasion, we have $(D_{ij}.F_9) = -1$. We make further replacement $D_{ij} \mapsto D_{ij} - F_9$ for those cases.

(2) Rewrite D_{ij} in terms of the $\mathbb{Z}$-basis $\{p^*H, F_1, \ldots, F_9, E_1, E_2, E_3\}$ so that D_{ij} is expressed in the following form:
$$D_{ij} = p^*(dH) - \bigl(\text{sum of exceptional curves}\bigr).$$

(3) Assume $h^0(D_{ij}) > 0$, then there exists an effective divisor D which is linearly equivalent to D_{ij}. Consider the plane curve p_*D. It is a plane curve of degree d which imposes several conditions corresponding to the negative part. Let $\mathcal{I}_{\mathrm{C}} \subset \mathcal{O}_{\mathbb{P}^2}$ be the ideal sheaf associated with the imposed conditions on p_*D. Compute $h^0(\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{I}_{\mathrm{C}})$. This number gives an upper bound of $h^0(D_{ij})$ (it is clear that if D' is an effective divisor linearly equivalent to D, such that p_*D and p_*D' coincide as plane curves, then D and D' must be the same curve in Y).

(4) As in Step 3(4), we will see that all the upper bound $h^0(D_{ij})$ fit in to the numerical invariant $\chi(-G^{\mathrm{g}}_i + G^{\mathrm{g}}_j)$. This shows that the upper bound $h^0(D_{ij})$ obtained in (3) exactly determines the three numbers $\{h^p_{ij} : p = 0, 1, 2\}$.

Step 6. As explained in Remark 5.8, the value $h^0(D_{ij})$ might depend on the configuration of p_*C_1 and p_*C_2. However, for general nodal cubics $p_*C_1 = (h_1 = 0)$, $p_*C_2 = (h_2 = 0)$, the minimum value of $h^0(D_{ij})$ is attained. This can be observed in the following way. Let $h = \sum_\alpha a_\alpha \mathbf{x}^\alpha$ be a homogeneous equation of degree d, where $\alpha = (\alpha_x, \alpha_y, \alpha_z)$ is the 3-tuple with $\alpha_x + \alpha_y + \alpha_z = d$ and $\mathbf{x}^\alpha = x^{\alpha_x} y^{\alpha_y} z^{\alpha_z}$. Then the ideal $\mathcal{I}_C$ impose linear relations on $\{a_\alpha\}_\alpha$, thus we get a linear system, namely a matrix M, with the variables $\{a_\alpha\}_\alpha$. After perturb h_1 and h_2 slightly, the rank of M would not decrease since $\{\operatorname{rank} M \geq r_0\}$ is an open condition for any fixed r_0. From this we get: if $h^0(D_{ij}) \leq r$ for at least one pair of p_*C_1 and p_*C_2, then $h^0(D_{ij}) \leq r$ for general p_*C_1 and p_*C_2.

Step 7. Let $h_1 = (y - z)^2 z - x^3 - x^2 z$, and $h_2 = x^3 - 2xy^2 + 2xyz + y^2 z$. These equations define plane nodal cubics such that

(1) p_*C_1 has the node at $[0, 1, 1]$, and p_*C_2 has the node at $[0, 0, 1]$;

(2) p_*C_2 has two tangent directions ($y = 0$ and $y = -2x$) at nodes;

(3) $p_*C_1 \cap p_*C_2$ contains two $\mathbb{Q}$-rational points, namely $[0, 1, 0]$ and $[-1, 1, 1]$.

We take $y = 0$ as the distinguished tangent direction at the node of p_*C_2, and take $p_*F_9 = [0, 1, 0]$, $p_*F_8 = [-1, 1, 1]$. The following ideals are the building blocks of the ideal $\mathcal{I}_C$ introduced in Step 3(3).

symbol	ideal form	ideal sheaf at the ...	divisor on Y
$\mathcal{I}_{E_1}$	$(x, y-z)$	node of p_*C_1	$-E_1$
$\mathcal{I}_{E_2+E_3}$	(x, y)	node of p_*C_2	$-(E_2+E_3)$
$\mathcal{I}_{E_2+2E_3}$	(x^2, y)	distinguished tangent at the node of p_*C_2	$-(E_2+2E_3)$
$\mathcal{J}_9$	(h_1, h_2)	nine base points	$-\sum_{i\le 9} F_i$
$\mathcal{J}_7$	$\mathcal{J}_9 / (x+z,y-z)(x,z)$	seven base points	$-\sum_{i\le 7} F_i$
$\mathcal{J}_8$	$(x+z, y-z)\mathcal{J}_7$	eight base points	$-\sum_{i\le 8} F_i$

Table 5.5

Note that the nine base points contain $[0, 1, 0]$ and $[-1, 1, 1]$, thus there exists an ideal $\mathcal{J}_7$ such that $\mathcal{J}_9 = (x+z, y-z)(x, z)\mathcal{J}_7$.

Step 8. We sketch the proof of $h^p_{10,9} = h^p(-G^{\mathrm{g}}_{10} + G^{\mathrm{g}}_9) = 0$, which illustrates several subtleties. Since $h^2_{10,9} = 0$ by Step 1, we only have to prove $h^0_{10,9} = 0$. Thus, we take $D^{\mathrm{g}}_{10,9} := -G^{\mathrm{g}}_{10} + G^{\mathrm{g}}_9$. As in Step 3(2), take $D'_{10,9} = p^*(3H) + 3F_9 - 2C_0 + 2C_1 - E_1 - (C_2 + E_2 + E_3) + 2(2C_2 + E_2)$. We have $(D'_{10,9}.C_2) = 0$, and $(D'_{10,9} - C_2 - E_2 . F_9) = -1$. Let $D_{10,9} := D'_{10,9} - C_2 - E_2 - F_9$. Then, $h^0(D_{10,9}) = h^0(D'_{10,9}) \ge h^0(D^{\mathrm{g}}_{10,9})$. As in Step 5(2), the divisor $D_{10,9}$ can be rewritten as

$$D_{10,9} = p^*(9H) - 2\sum_{i=1}^{8} F_i - 5E_1 - 4E_2 - 7E_3.$$

Since $\mathcal{I}^2_{E_2+E_3}$ imposes more conditions than $\mathcal{I}_{E_2+2E_3}$, the ideal of (minimal) conditions corresponding to $-4E_2 - 7E_3$ is $\mathcal{I}_{E_2+E_3} \cdot \mathcal{I}^3_{E_2+2E_3}$. Thus, the plane curve $p_*D_{10,0}$ corresponds to a nonzero section of

$$H^0(\mathcal{O}_{\mathbb{P}^2}(9) \otimes \mathcal{J}_8^2 \cdot \mathcal{I}^5_{E_1} \cdot \mathcal{I}_{E_2+E_3} \cdot \mathcal{I}^3_{E_2+2E_3}).$$

We ask Macaulay 2 to find the rank of this group, and the result is zero. This can be found in `ExcColl_Dolgachev.m2` [5]. In simiarly ways, we obtain the following table (be aware of the difference with Table 5.3).

	G^{g}_0	G^{g}_8	G^{g}_9	G^{g}_{10}	G^{g}_{11}
G^{g}_0	1 0 0	0 0 1	0 0 1	0 0 3	0 0 6
G^{g}_8		1 0 0		0 0 2	0 0 5
G^{g}_9			1 0 0	0 0 2	0 0 5
G^{g}_{10}				1 0 0	0 0 3
G^{g}_{11}					1 0 0

Table 5.6

The table in below gives a short summary on the computations done in `ExcColl_Dolgachev.m2` [5].

(i,j)	result	choice of D_{ij}
$(9,0)$	$h^0_{9,0}=0$	$p^*(5H)-\sum_{i\le 9}F_i-3E_1-2E_2-4E_3$
$(10,0)$	$h^0_{10,0}=0$	$p^*(14H)-3\sum_{i\le 8}F_i-8E_1-6E_2-11E_3$
$(10,8)$	$h^0_{10,8}=0$	$p^*(9H)-2\sum_{i\le 7}F_i-F_8-6E_1-3E_2-6E_3$
$(10,9)$	$h^0_{10,9}=0$	$p^*(9H)-2\sum_{i\le 8}F_i-5E_1-4E_2-7E_3$
$(11,0)$	$h^0_{11,0}=0$	$p^*(31H)-7\sum_{i\le 8}F_i-F_9-18E_1-11E_2-22E_3$
$(11,8)$	$h^0_{11,8}=0$	$p^*(26H)-6\sum_{i\le 7}F_i-5F_8-F_9-14E_1-10E_2-20E_3$
$(11,9)$	$h^0_{11,9}=0$	$p^*(26H)-6\sum_{i\le 8}F_i-15E_1-9E_2-18E_3$
$(0,10)$	$h^2_{0,10}=3$	$p^*(17H)-4\sum_{i\le 8}F_i-F_9-9E_1-6E_2-12E_3$
$(0,11)$	$h^2_{0,11}=6$	$p^*(31H)-7\sum_{i\le 8}F_i-F_9-17E_1-13E_2-23E_3$
$(8,11)$	$h^2_{8,11}=5$	$p^*(26H)-6\sum_{i\le 7}F_i-5F_8-F_9-15E_1-9E_2-18E_3$
$(9,11)$	$h^2_{9,11}=5$	$p^*(26H)-6\sum_{i\le 8}F_i-14E_1-10E_2-19E_3$

Table 5.7

Note that the numbers $h^p_{11,10}$ and $h^p_{10,11}$ are free to evaluate; indeed, $-G^{\mathrm{g}}_{11}+G^{\mathrm{g}}_{10}=-G^{\mathrm{g}}_{10}$, thus $h^p_{11,10}=h^p_{10,0}$ and $h^p_{10,11}=h^p_{0,10}$. Finally, perturb the cubics p_*C_1 and p_*C_2 so that Table 5.6 remains valid and Lemma 5.9 is applicable. Then, Table 5.3 is verified immediately. □

Remark 5.8. Assume that the nodal curves p_*C_1, p_*C_2 are in a special position so that the node of p_*C_1 is located on the distinguished tangent line at the node of p_*C_2. Then, the proper transform ℓ of the unique line through the nodes of p_*C_1 and p_*C_2 has the following divisor expression:

$$\ell = p^*H - E_1 - (E_2 + 2E_3).$$

In particular, the divisor $D_{90}=p^*(5H)-\sum_{i\le 9}F_i-3E_1-2(E_2+2E_3)$ is linearly equivalent to $2\ell+C_1+E_1$, thus $h^0(D_{90})>0$. Consequently, for this particular configuration of p_*C_1 and p_*C_2, we cannot prove $h^0_{90}=0$ using upper-semicontinuity. However, the numerical method (Step 3 in the proof of the previous theorem) cannot detect such variances originated from the position of nodal cubics, hence it cannot be applied to the proof of $h^0_{90}=0$

The following lemma, used in the end of the proof of Theorem 5.7, illustrates the symmetric nature of $F_1,\dots,F_8$.

Lemma 5.9. *Assume that X^{g} is originated from a cubic pencil generated by two general plane nodal cubics p_*C_1 and p_*C_2. Let $D\in \operatorname{Pic} Y$ be a divisor on the rational elliptic surface Y. Assume that in the expression of D in terms of $\mathbb{Z}$-basis $\{p^*H,F_1,\dots,F_9,E_1,E_2,E_3\}$, the coefficients of $F_1,\dots,F_8$ are same. Then, $h^p(D+F_i)=h^p(D+F_j)$ for any $p\ge 0$ and $1\le i,j\le 8$.*

Proof. Since $\operatorname{Aut}\mathbb{P}^2=\mathrm{PGL}(3,\mathbb{C})$ sends arbitrary 4 points (of which any three are not colinear) to arbitrary 4 points (of which any three are not colinear). Using projective linear equivalences, we may assume the following.

(a) The base point p_*F_9 is $\mathbb{Q}$-rational.
(b) The nodes of p_*C_1 and p_*C_2 are $\mathbb{Q}$-rational.
(c) The distinguished tangent direction at the node of p_*C_2 is defined over $\mathbb{Q}$.

Also, we may take nodal cubics p_*C_1, p_*C_2 which satisfy the further assumptions:

(d) The ideals of p_*C_1, p_*C_2 are defined over $\mathbb{Q}$.[4)]
(e) The eight points $p_*F_1, \dots, p_*F_8$ are contained in the affine space $(z \neq 0) \subset \mathbb{P}^2_{x,y,z}$.
(f) Let $p_*F_i = [\alpha_i, \beta_i, 1] \in \mathbb{P}^2$ for $\alpha_i, \beta_i \in \mathbb{C}$, and let $H_\alpha \in \mathbb{Q}[t]$ (resp. $H_\beta \in \mathbb{Q}[t]$) be the irreducible polynomial having α_1 (resp. β_1) as a root. Then, both H_α and H_β are of degrees 8, and $H_\alpha \neq H_\beta$ up to multiplication by $\mathbb{Q}^\times$.

The last condition can be interpreted by resultants. Let $h_i \in \mathbb{Q}[x,y,z]$ be the defining equation of p_*C_i. Let $\operatorname{res}(h_1,h_2;x)$ be the resultant of $h_1(x,y,1), h_2(x,y,1)$ regarded as elements of $(\mathbb{Q}[y])[x]$. The polynomial $\operatorname{res}(h_1,h_2;x) \in \mathbb{Q}[y]$ reads the y-coordinates of the base points, thus $\operatorname{res}(h_1,h_2;x) =$ (linear or constant term) $\times H_\beta$. Note that the linear or constant term always appears due to (a). The condition (f) imposes an open condition on plane nodal cubics; after locating p_*F_9 at a $\mathbb{Q}$-rational point by $\mathrm{PGL}(3,\mathbb{C})$ action, the degree 8 factor of $\operatorname{res}(h_1,h_2;x)$ (resp. $\operatorname{res}(h_1,h_2;y)$), corresponding to $p_*F_1, \dots, p_*F_8$, is irreducible for general h_1, h_2.

If p_*C_1 and p_*C_2 satisfy the conditions (a)–(f) above, the blow up construction $p\colon Y \to \mathbb{P}^2$ is well-defined over $\mathbb{Q}$, thus there exists a variety $Y_\mathbb{Q}$ over $\mathbb{Q}$ such that $Y = Y_\mathbb{Q} \times_\mathbb{Q} \mathbb{C}$. Let $\tau \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$ be a field automorphism fixing $\mathbb{Q}$, and mapping α_i to α_j. Then τ induces an automorphism of $\mathbb{P}^2$ which fixes p_*C_1 and p_*C_2 by (d). It follows that $[\alpha_j, \tau(\beta_i), 1]$ is one of the nine base points $\{p_*F_i\}$. Since H_α and H_β are different, there is no point of the form $[\alpha_j, \beta_k, 1]$ among the nine base points except when $k = j$. It follows that $\tau(\beta_i) = \beta_j$. Let $\tau_Y := \mathrm{id}_{Y_\mathbb{Q}} \times \tau$ be the automorphism of $Y = Y_\mathbb{Q} \times_\mathbb{Q} \mathbb{C}$. According to our assumptions, it satisfies the following properties:

(1) τ_Y fixes F_9, E_1, E_2, E_3;
(2) τ_Y permutes $F_1, \dots, F_8$;
(3) τ_Y maps F_i to F_j.

Furthermore, since the coefficients of $F_1, \dots, F_8$ are same in the expression of D, τ_Y fixes D. It follows that $\tau_Y^*\colon \operatorname{Pic} Y \to \operatorname{Pic} Y$ maps $D + F_j$ to $D + F_i$. In particular, $H^p(D+F_j) = H^p(\tau_Y^*(D+F_i)) \simeq H^p(D+F_i)$. □

5.2. **Incompleteness of the collection.** Let $\mathcal{A} \subset \mathrm{D}^\mathrm{b}(X^\mathrm{g})$ be the orthogonal subcategory

$$\big\langle \mathcal{O}_{X^\mathrm{g}}(G_0^\mathrm{g}),\ \mathcal{O}_{X^\mathrm{g}}(G_1^\mathrm{g}),\ \dots,\ \mathcal{O}_{X^\mathrm{g}}(G_{11}^\mathrm{g}) \big\rangle^\perp,$$

so that there exists a semiorthogonal decomposition

$$\mathrm{D}^\mathrm{b}(X^\mathrm{g}) = \big\langle \mathcal{A},\ \mathcal{O}_{X^\mathrm{g}}(G_0^\mathrm{g}),\ \mathcal{O}_{X^\mathrm{g}}(G_1^\mathrm{g}),\ \dots,\ \mathcal{O}_{X^\mathrm{g}}(G_{11}^\mathrm{g}) \big\rangle.$$

We will prove that $K_0(\mathcal{A}) = 0$, $\mathrm{HH}_\bullet(\mathcal{A}) = 0$, but $\mathcal{A} \not\simeq 0$. Such a category is called a *phantom* category. To give a proof, we claim that the *pseudoheight* of the collection (5.14) is at least 2. Once we achieve the claim, [19, Corollary 4.6] implies that $\mathrm{HH}^0(\mathcal{A}) \simeq \mathrm{HH}^0(X^\mathrm{g}) = \mathbb{C}$, thus $\mathcal{A} \not\simeq 0$.

Definition 5.10.

(a) Let E_1, E_2 be objects in $\mathrm{D}^\mathrm{b}(X^\mathrm{g})$. The *relative height* $e(E_1,E_2)$ is the minimum of the set

$$\{p : \operatorname{Hom}(E_1, E_2[p]) \neq 0\} \cup \{\infty\}.$$

4)Note that the space $(\mathbb{P}^9_\mathbb{Q})^*$ of plane cubic curves over $\mathbb{Q}$ is Zariski dense in $(\mathbb{P}^9_\mathbb{C})^*$.

(b) Let $\langle F_0, \ldots, F_m \rangle$ be an exceptional collection in $\mathrm{D^b}(X^{\mathrm{g}})$. The *anticanonical pseudoheight* is defined by
$$\mathrm{ph}_{\mathrm{ac}}(F_0, \ldots, F_m) = \min\Big(\sum_{i=1}^{p} e(F_{a_{i-1}}, F_{a_i}) + e(F_{a_p}, F_{a_0} \otimes \mathcal{O}_{X^{\mathrm{g}}}(-K_{X^{\mathrm{g}}})) - p\Big),$$
where the minimum is taken over all possible tuples $0 \le a_0 < \ldots < a_p \le m$.

The pseudoheight is given by the formula $\mathrm{ph}(F_0, \ldots, F_m) = \mathrm{ph}_{\mathrm{ac}}(F_0, \ldots, F_m) + \dim X^{\mathrm{g}}$, thus it suffices to prove that $\mathrm{ph}_{\mathrm{ac}}(G_0^{\mathrm{g}}, \ldots, G_{11}^{\mathrm{g}}) \ge 0$. Computing the exact value of $\mathrm{ph}_{\mathrm{ac}}(G_0^{\mathrm{g}}, \ldots, G_{11}^{\mathrm{g}})$ needs more works, however just proving its nonnegativity is an immediate consequence of Theorem 5.7.

Corollary 5.11. *In the semiorthogonal decomposition*
$$\mathrm{D^b}(X^{\mathrm{g}}) = \big\langle \mathcal{A},\ \mathcal{O}_{X^{\mathrm{g}}}(G_0^{\mathrm{g}}),\ \ldots,\ \mathcal{O}_{X^{\mathrm{g}}}(G_{11}^{\mathrm{g}}) \big\rangle,$$
we have $K_0(\mathcal{A}) = 0$*,* $\mathrm{HH}_\bullet(\mathcal{A}) = 0$*, and* $\mathrm{HH}^0(\mathcal{A}) = \mathbb{C}$.

Proof. Since X^{g} is a surface of special type, the Bloch conjecture holds for X^{g} [30, §11.1.3]. Thus the Grothendieck group $K_0(X^{\mathrm{g}})$ is a free abelian group of rank 12 (see for e.g. [9, Lemma 2.7]). Furthermore, Hochschild-Kostant-Rosenberg isomorphism for Hochschild homology says
$$\mathrm{HH}_k(X^{\mathrm{g}}) \simeq \bigoplus_{q-p=k} H^{p,q}(X^{\mathrm{g}}),$$
hence, $\mathrm{HH}_\bullet(X^{\mathrm{g}}) \simeq \mathbb{C}^{\oplus 12}$. It is well-known that K_0 and $\mathrm{HH}_\bullet$ are additive invariants with respect to semiorthogonal decompositions, thus $K_0(X^{\mathrm{g}}) \simeq K_0(\mathcal{A}) \oplus K_0({}^\perp\mathcal{A})$, and $\mathrm{HH}_\bullet(X^{\mathrm{g}}) = \mathrm{HH}_\bullet(\mathcal{A}) \oplus \mathrm{HH}_\bullet({}^\perp\mathcal{A})$.[5)] If E is an exceptional vector bundle, then $\mathrm{D^b}(\langle E \rangle) \simeq \mathrm{D^b}(\operatorname{Spec}\mathbb{C})$ as $\mathbb{C}$-linear triangulated categories, thus $K_0({}^\perp\mathcal{A}) \simeq \mathbb{Z}^{\oplus 12}$ and $\mathrm{HH}_\bullet({}^\perp\mathcal{A}) \simeq \mathbb{C}^{\oplus 12}$. It follows that $K_0(\mathcal{A}) = 0$ and $\mathrm{HH}_\bullet(\mathcal{A}) = 0$.

For any $0 \le j < i \le 11$,
$$e(G_j^{\mathrm{g}}, G_i^{\mathrm{g}}) = \begin{cases} \infty & \text{if } 1 \le j < i \le 9 \\ 2 & \text{otherwise} \end{cases}$$
by Theorem 5.7. Thus, for any length p chain $0 \le a_0 < \ldots, a_p \le 11$,
$$e(G_{a_0}^{\mathrm{g}}, G_{a_1}^{\mathrm{g}}) + \ldots + e(G_{a_{p-1}}^{\mathrm{g}}, G_{a_p}^{\mathrm{g}}) + e(G_{a_p}, G_{a_0} - K_{X^{\mathrm{g}}}) - p \ge p,$$
which shows that $\mathrm{ph}_{\mathrm{ac}}(G_0^{\mathrm{g}}, \ldots, G_{11}^{\mathrm{g}}) \ge 0$. By [19, Corollary 4.6], $\mathrm{HH}^0(\mathcal{A}) \simeq \mathrm{HH}^0(X^{\mathrm{g}}) \simeq \mathbb{C}$. □

5.3. **Cohomology computations.** We finish with the Dictionary 5.12 of cohomology computations that appeared in the proof of Theorem 5.7. Given a divisor D, the main strategy is that we take various nonsingular curves $A_1, \ldots, A_r$ such that the values $(D.A_1)$, $(D - A_1 \,.\, A_2)$, $(D - A_1 - A_2 \,.\, A_3)$, $\ldots$ are small. The algorithm how these curves compute $h^0(D)$ will be presented in Dictionary 5.12.

Most of the curves $A_1, \ldots, A_r$ (not necessarily distinct) will be chosen among the divisors illustrated in Figure 2.1, but we have to implement one more curve, which did not appear in Figure 2.1. Let ℓ be the proper transform of the unique line in $\mathbb{P}^2$ passing through the nodes of p_*C_1 and p_*C_2. In the divisor form,
$$\ell = p^*H - E_1 - (E_2 + E_3).$$

5) By definition of $\mathcal{A}$, ${}^\perp\mathcal{A}$ is the smallest full triangulated subcategory containing the collection (5.14) in Theorem 5.7.

Due to the divisor forms

$$
\begin{aligned}
C_1 &= p^*(3H) - 2E_1 - \textstyle\sum_{i=1}^9 F_i,\\
C_2 &= p^*(3H) - (2E_2 + 3E_3) - \textstyle\sum_{i=1}^9 F_i, \text{ and}\\
C_0 &= p^*(3H) - \textstyle\sum_{i=1}^9 F_i,
\end{aligned}
$$

it is straightforward to write down the intersections involving ℓ.

	p^*H	F_i	C_0	C_1	E_1	C_2	E_2	E_3	ℓ
ℓ	1	0	3	1	1	1	1	0	-1

Table 5.8

Dictionary 5.12. For each of the following Cartier divisors on Y, we give upper bounds of h^0. We take smooth rational curves $A_1, \ldots, A_r$, and consider the exact sequence

$$0 \to H^0(D - S_i) \to H^0(D - S_{i-1}) \to H^0(\mathcal{O}_{\mathbb{P}^1}(D - S_{i-1})),$$

where $S_i = \sum_{j \le i} A_j$. This gives the inequality $h^0(D - S_{i-1}) \le h^0(D - S_i) + h^0((D - S_i)\big|_{\mathbb{P}^1})$. Inductively, we get

$$h^0(D) \le h^0(D - S_r) + \sum_{i=1}^{r-1} h^0((D - S_i)\big|_{\mathbb{P}^1}). \tag{5.15}$$

In what follows, we take $A_1, \ldots, A_r$ carefully so that $h^0(D - S_r) = 0$, and that the values $h^0(D - S_i\big|_{\mathbb{P}^1})$ are as small as possible. In each item in the dictionary, we first present the target divisor D and the bound of $h^0(D)$. After then, we give a list of smooth rational curves in the following format:

$$A_1,\ A_2,\ \ldots,\ A_i{}^{(\checkmark)\times d},\ \ldots,\ n \times A_j,\ A_{j+n}, \ldots, A_r.$$

The symbol $(\checkmark) \times d\,(d \ge 1)$ indicates the situation that we have $(D - S_{i-1} \,.\, A_i) = d - 1$. In those cases, the right hand side of (5.15) increases by d. We omit "$\times d$" if $d = 1$. Also, $n \times A_j$ means that the same curve appears n times in the list. In other words, it indicates the case $A_j = A_{j+1} = \ldots A_{j+n-1}$. We conclude by showing that $D - S_r$ is not an effective divisor. The upper bound of $h^0(D)$ will be given by the number of $(\checkmark)$ in the list. Since all of these calculations are routine, we omit the details. From now on, i is a number between $1, 2, \ldots, 8$.

(1) $D = p^*(2H) + F_9 - F_i - 2C_0 + C_1 + C_2 + E_2 + E_3$ $\qquad h^0(D) = 0$
The following is the list of curves $A_1, \ldots, A_r$ (the order is important): F_9, ℓ, E_2, ℓ. The resulting divisor is

$$D - A_1 - \ldots - A_r = p^*(2H) - F_i - 2C_0 + C_1 + C_2 + E_3 - 2\ell.$$

Since $\ell = p^*H - E_1 - (E_2 + E_3)$ and $C_0 = C_1 + 2E_1 = C_2 + 2E_2 + 3E_3$, $D - A_1 - \ldots - A_r = -F_i$. It follows that $H^0(D) \simeq H^0(-F_i) = 0$.

(2) $D = p^*(2H) + F_9 - F_i - C_0 + C_1 - E_1 + 2C_2 + E_2.$ $\qquad h^0(D) \le 1$
Rule out C_2, E_2, $\ell^{(\checkmark)}$, C_1, F_9, C_2, ℓ, E_1. The resulting divisor is $p^*(2H) - F_i - C_0 - 2E_1 - 2\ell = -F_i - C_2 - E_3$. Since there is only one checkmark, $h^0(D) \le h^0(-F_i - C_2 - E_3) + 1 = 1$.

(3) $D = p^*(2H) - 2C_0 + C_1 + C_2 + E_2 + E_3$ $\qquad h^0(D) \le 1$
Rule out ℓ, E_2, ℓ, $C_2{}^{(\checkmark)}$. The remaining part is $p^*(2H) - 2C_0 + C_1 + E_3 - 2\ell = -C_2$, thus $h^0(D) \le 1$.

(4) $D = p^*(3H) + 2F_9 + F_i - 2C_0 + 2C_1 - E_1 + 3C_2 + E_2 - E_3$ $\qquad h^0(D) \le 2$
The following is the list of divisors that we have to remove:

$$C_2,\ E_2,\ \ell^{(\checkmark)},\ E_2,\ F_9^{(\checkmark)},\ C_2,\ E_2,\ \ell,\ C_1,\ F_9,\ F_i,\ \ell.$$

The remaining part is $p^*(3H) - 2C_0 + C_1 - E_1 + C_2 - E_2 - E_3 - 3\ell = -E_3$, thus $h^0(D) \le 2$.

(5) $D = p^*(3H) + 3F_9 - 3C_0 + 3C_1 + 3C_2 + 2E_2 + E_3$ $\qquad h^0(D) \le 2$
Rule out the following curves:

$$F_9^{(\checkmark)},\ C_1,\ C_2,\ E_2,\ F_9,\ \ell,\ E_2^{(\checkmark)},\ \ell,\ C_2,\ \ell,\ E_2,\ E_3,\ F_9,\ C_1,\ E_1.$$

The remaining part is $p^*(3H) - 3C_0 + C_1 - E_1 + C_2 - E_2 - 3\ell = -C_0$, thus $h^0(D) \le 2$.

6. Appendix

6.1. **A brief review on Hacking's construction.** Let $n > a > 0$ be coprime integers, let X be a projective normal surface with quotient singularities, and let $(P \in X)$ be a T_1-singularity of type $(0 \in \mathbb{A}^2/\frac{1}{n^2}(1, na-1))$. Suppose there exists a one parameter deformation $\mathcal{X}/(0 \in T)$ of X over a smooth curve germ $(0 \in T)$ such that $(P \in \mathcal{X})/(0 \in T)$ is a $\mathbb{Q}$-Gorenstein smoothing of $(P \in X)$.

Proposition 6.1 ([11, §3])**.** *Take a base extension $(0 \in T') \to (0 \in T)$ of ramification index a, and let $\mathcal{X}'$ be the pull back along the extension. Then, there exists a proper birational morphism $\Phi\colon \tilde{\mathcal{X}} \to \mathcal{X}'$ satisfying the following properties.*

(a) *The central fiber $W = \Phi^{-1}(P)$ is a projective normal surface isomorphic to*

$$(xy = z^n + t^a) \subset \mathbb{P}_{x,y,z,t}(1, na-1, a, n).$$

(b) *The morphism Φ is an isomorphism outside W.*

(c) *The central fiber $\tilde{\mathcal{X}}_0 = \Phi^{-1}(\mathcal{X}'_0)$ is reduced and has two irreducible components: $\tilde{X}_0$ the proper transform of X, and W. The intersection $Z := \tilde{X}_0 \cap W$ is a smooth rational curve given by $(t = 0)$ in W. Furthermore, the surface $\tilde{X}_0$ can be obtained in the following way: take a minimal resolution $Y \to X$ of $P \in X$, and let $G_1, \dots, G_r$ be the chain of exceptional curves arranged so that $(G_i.G_{i+1}) = 1$ and $(G_r^2) = -2$. Then the contraction of $G_2, \dots, G_r$ defines $\tilde{X}_0$. Clearly, G_1 maps to Z along the contraction $Y \to \tilde{X}_0$.*

Proposition 6.2 ([11, Proposition 5.1])**.** *There exists an exceptional vector bundle G of rank n on W such that $G\big|_Z \simeq \mathcal{O}_Z(1)^{\oplus n}$.*

Remark 6.3. Note that in the decomposition $\tilde{\mathcal{X}}_0 = \tilde{X}_0 \cup W$, the surface W is completely determined by the type of singularity $(P \in X)$, whereas $\tilde{X}_0$ reflects the global geometry of X. In some circumstances, W and G have explicit descriptions.

(a) Suppose $a = 1$. In $\mathbb{P}_{x,y,z,t}(1, n-1, 1, n)$, $W_2 = (xy = z^n + t)$ and $Z_2 = (xy = z^n, t = 0)$ by Proposition 6.1. The projection map $\mathbb{P}_{x,y,z,t}(1, n-1, 1, n) \dashrightarrow \mathbb{P}_{x,y,z}(1, n-1, 1)$ sends W_2 isomorphically onto $\mathbb{P}_{x,y,z}$, thus we get

$$W_2 = \mathbb{P}_{x,y,z}(1, n-1, 1), \quad \text{and} \quad Z_2 = (xy = z^n) \subset \mathbb{P}_{x,y,z}(1, n-1, 1).$$

(b) Suppose $(n, a) = (2, 1)$, then it can be shown (by following the proof of Proposition 6.2) that $W = \mathbb{P}^2_{x,y,z}$, $G = \mathcal{T}_{\mathbb{P}^2}(-1)$ where $\mathcal{T}_{\mathbb{P}^2} = (\Omega^1_{\mathbb{P}^2})^\vee$ is the tangent sheaf of the plane. Moreover, the smooth rational curve $Z = \tilde{X}_0 \cap W$ is embedded as a smooth conic $(xy = z^2)$ in W.

The final proposition would present how to obtain an exceptional vector bundle on a general fiber of the smoothing.

Proposition 6.4 ([11, §4]). *Let X^{g} be a general fiber of the deformation $\mathcal{X}/(0 \in T)$, and assume $H^2(\mathcal{O}_{X^{\mathrm{g}}}) = H^1(X^{\mathrm{g}}, \mathbb{Z}) = 0$.*[6] *Let G be an exceptional vector bundle on W as in Proposition 6.2. Suppose there exists a Weil divisor $D \in \operatorname{Cl} X$ such that D does not pass through the singular points of X except P, and the proper transform $D' \subset \tilde{X}_0$ of X satisfies $(D'.Z) = 1$ and $\operatorname{Supp} D' \subset \tilde{X}_0 \setminus \operatorname{Sing} \tilde{X}_0$. Then the vector bundles $\mathcal{O}_{\tilde{X}_0}(D')^{\oplus n}$ and G glue along $\mathcal{O}_Z(1)^{\oplus n}$ to produce an exceptional vector bundle $\tilde{E}$ on $\tilde{\mathcal{X}}_0$. Furthermore, the vector bundle $\tilde{E}$ deforms uniquely to an exceptional vector bundle $\tilde{\mathcal{E}}$ on $\tilde{\mathcal{X}}$. Restriction $\tilde{\mathcal{E}}\big|_{X^{\mathrm{g}}}$ to the general fiber is an exceptional vector bundle on X^{g} of rank n.*

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